MATH 281 (PHILLIPS): SOLUTIONS TO MIDTERM 2 (CORRECTED AND EXPANDED)

Problem 1 (1 point)**.** The Squeeze Theorem was invented for the purpose of:
(a) Torturing calculus students.
(b) Torturing functions.
(c) Torturing calculus professors who have to teach it.

Solution. Personally, I think (b) is the best answer. After all, it is the function which gets squeezed. As one student in a previous class said in his answer, "The other two are just a side benefit." □

Problem 2 (15 points)**.** Find an equation for the normal plane to the parametric curve given by $\mathbf{r}(t) = \big\langle \cos(t-1),\ 1/t,\ -2t \big\rangle$ at the point $(1,\ 1,\ -2)$.

Read the problem! It did not ask for the normal *vector*, and it did not ask for the equation of the tangent line.

Solution. We have $\mathbf{r}(t) = (1,\ 1,\ -2)$ when $t = 1$. Start by calculating:

$$\mathbf{r}'(t) = \big\langle -\sin(t-1),\ -1/t^2,\ -2 \big\rangle.$$

So $\mathbf{r}'(1) = \langle 0,\ -1,\ -2\rangle$. Take this to be the normal vector of the plane. Then the equation of the plane is

$$\langle 0,\ -1,\ -2\rangle \cdot (\langle x,\ y,\ z\rangle - \langle 1,\ 1,\ -2\rangle) = 0.$$

This is $-(y-1) - 2(z+2) = 0$, which simplifies to $-y - 2z - 3 = 0$, or to $y + 2z + 3 = 0$. (The simplification is *required.*) □

Problem 3 (12 points)**.** Let

$$h(x, y, z) = 3x^{65} + xy\sin(y)\cos(z) + \arctan(z^{2020}) + \pi^2.$$

Find $D_2h(x, y, z)$ (in the book usually called $h_y(x, y, z)$).

Solution. All summands except $xy\sin(y)\cos(z)$ are independent of y, so their partial derivatives with respect to y are zero. In the remaining summand, x and $\cos(z)$ are constants when differentiating with respect to y, so we get

$$D_2h(x, y, z) = \frac{\partial}{\partial y}\big(xy\sin(y)\cos(z)\big) = x\left(\frac{d}{dy}\big(y\sin(y)\big)\right)\cos(z) = x\big(\sin(y) + y\cos(y)\big)\cos(z).$$

There is no need to multiply this out. □

Here is a version of the solution showing more steps as a substitute for the explanation part of the first solution.

Alternate solution. We have

$$\begin{aligned}\frac{\partial h}{\partial y}(x, y, z) &= \frac{\partial}{\partial y}\big(3x^{65}\big) + \frac{\partial}{\partial y}\big(xy\sin(y)\cos(z)\big) + \frac{\partial}{\partial y}\big(\arctan(z^{2020})\big) + \frac{\partial}{\partial y}\big(\pi^2\big) \\ &= 0 + x\left(\frac{d}{dy}\big(y\sin(y)\big)\right)\cos(z) + 0 + 0 \\ &= x\big(\sin(y) + y\cos(y)\big)\cos(z).\end{aligned}$$

This completes the solution. □

Date: 27 November 2020.

Problem 4 (10 points)**.** Find the domain of the function $f(x,y) = \sqrt{x} - \ln(y+1)$. Give reasons.

Solution. The domain consists of all pairs (x,y) in $\mathbb{R}^2$ at which the argument of the square root is nonnegative and the argument of the logarithm is strictly positive, that is, $x \geq 0$ and $y > -1$. That is, the domain is

$$\{(x,y) \colon x \geq 0 \text{ and } y > -1\} \qquad \text{or} \qquad \{(x,y) \,|\, x \geq 0 \text{ and } y > -1\}.$$

(Both versions, with ":" and with "|", are standard notation.) □

A better version (which I have not seen in the book) is

$$\{(x,y) \in \mathbb{R}^2 \colon x \geq 0 \text{ and } y > -1\}.$$

It is not correct to write "~~x, y in~~ $\mathbb{R}^2$". ~~This~~ Similarly, is not correct to write

$$\{x, y \colon x \geq 0 \text{ and } y > -1\} \qquad \text{or} \qquad \{x, y \in \mathbb{R}^2 \colon x \geq 0 \text{ and } y > -1\}.$$

These answer will receive very little credit, because ~~it tries~~ they try to describe a subset of $\mathbb{R}$ rather than a subset of $\mathbb{R}^2$.

Also, writing "$x = [0, \infty)$" is wrong: x should be a real number, but this says x is equal to an interval.

Problem 5 (12 points)**.** A function f of two variables satisfies

$$f(7,6) = 11, \qquad f(7,0) = 2, \qquad f(0,6) = -3,$$

$$D_1 f(7,6) = -2, \qquad D_1 f(7,0) = 8, \qquad D_2 f(7,6) = 3, \qquad \text{and} \qquad D_2 f(0,6) = 15.$$

Use the linear approximation (tangent plane approximation) to estimate the number $f(6.95, 6.02)$. (You will not need to use all the information provided.)

Solution. The linear approximation at (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

(This is an *approximation*, not an equality, and therefore it is *not* correct to write "=" here.)

Take $a = 7$, $b = 6$, $x = 6.95$, and $y = 6.02$. This gives

$$\begin{aligned} f(6.95, 6.02) &\approx f(7,6) + f_x(7,6)(6.95 - 7) + f_y(7,6)(6.02 - 6) \\ &= 11 + (-2)(-0.05) + (3)(0.02) = 11.16. \end{aligned}$$

Thus, $f(6.95, 6.02) \approx 11.16$. (Again, it is *not* correct to write "=" here.) □

Problem 6 (12 points)**.** Let f be a differentiable function of two variables. (In the notation for partial derivatives, call them x and y.) Let $g(s,t) = f(\cos(t) - s,\ 2s^2 + 3t)$. Suppose

$$f(0,2) = 7, \qquad g(0,2) = 3, \qquad f_x(0,2) = -5, \qquad \text{and} \qquad f_y(0,2) = 2,$$

and

$$f(1,0) = 3, \qquad g(1,0) = 7, \qquad f_x(1,0) = 3, \qquad \text{and} \qquad f_y(1,0) = -4.$$

Find $g_s(1,0)$. (You will not need to use all the information provided.)

Restated with partial derivatives labelled by position (ignore this if the version above is clear): Suppose

$$f(0,2) = 7, \qquad g(0,2) = 3, \qquad D_1 f(0,2) = -5, \qquad \text{and} \qquad D_2 f(0,2) = 2,$$

and

$$f(1,0)=3, \qquad g(1,0)=7, \qquad D_1 f(1,0)=3, \qquad \text{and} \qquad D_2 f(1,0)=-4.$$

Find $D_1 g(1,0)$.

Solution. Set $x(s,t)=\cos(t)-s$ and $y(s,t)=2s^2+3t$. Then $x_s(s,t)=-1$ and $y_s(s,t)=4s$. Also $x(1,0)=\cos(0)-1=0$ and $y(1,0)=2$. So

$$\begin{aligned} g_s(1,0) &= f_x\big(x(1,0),\, y(1,0)\big)x_s(1,0) + f_y\big(x(1,0),\, y(1,0)\big)y_s(1,0) \\ &= f_x(0,2)x_s(1,0) + f_y(0,2)y_s(1,0) = (-5)(-1)+(2)(4)=13. \end{aligned}$$

The simplification $\cos(0)-1=0$ is *required.* □

To keep track of where to evaluate f_x and f_y, recall the one variable chain rule: $(f\circ g)'(x)=f'(g(x))g'(x)$, *not* ~~$f'(x)g'(x)$~~.

Problem 7 (16 points)**.** Let $f(x,y,z)=xy^2z^3+\sin(z)-x$. Find a unit vector in the direction in which f *decreases* the fastest at the point $(2,1,0)$. At what rate does f decrease in this direction?

Solution. To get some vector in the right direction, we start with $\mathbf{w}=\mathbf{grad}(f)(2,1,0)$. (This is the direction in which f *increases* the fastest. The direction of fastest decrease is the opposite direction.) Calculate

$$\mathbf{grad}(f)(x,y,z)=\langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\rangle = \left\langle y^2z^3-1,\ 2xyz^3,\ 3xy^2z^2+\cos(z)\right\rangle,$$

so

$$\mathbf{grad}(f)(2,1,0)=\langle -1,\ 0,\ 1\rangle.$$

To get the direction in which f *decreases* the fastest, use $\mathbf{v}=-\mathbf{w}=\langle 1,\ 0,\ -1\rangle$. We need a unit vector in this direction, which is

$$\left(\frac{1}{\|\mathbf{v}\|}\right)v=\left\langle \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right\rangle.$$

The rate of decrease is $\|\mathbf{w}\|=\sqrt{2}$. □

It isn't correct to say that the rate of decrease is $-\sqrt{2}$. Is *is* correct to say that the rate of *increase* is $-\sqrt{2}$.

Problem 8 (10 points)**.** Consider the surface in $\mathbb{R}^3$ given by

$$\frac{(x-3)^2}{4}+y^2+\frac{z^2}{16}=1.$$

Describe and draw its trace in the plane $z=\sqrt{12}$. In your graph, be sure to label the axes and put scales on the axes.

Solution. The trace in the plane $z=\sqrt{12}$ is given by the equation

$$\frac{(x-3)^2}{4}+y^2+\frac{12}{16}=1,$$

or

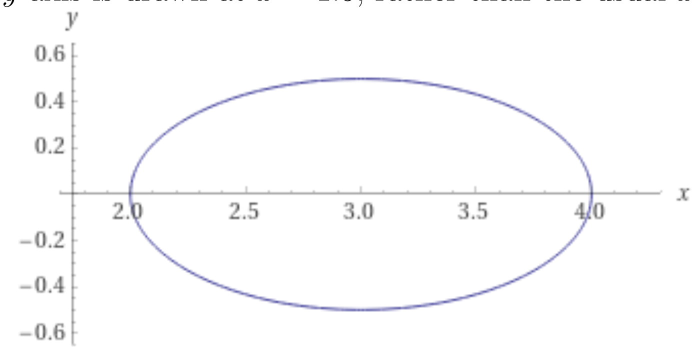
$$\frac{(x-3)^2}{4}+y^2=\frac{1}{4}.$$

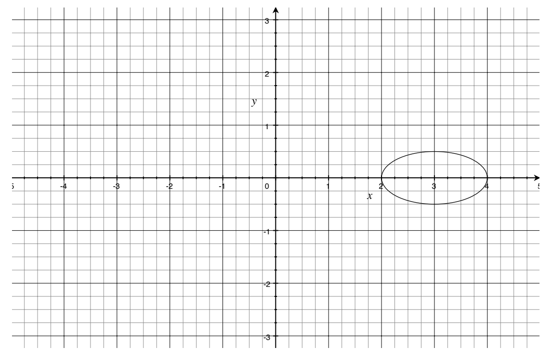
It is easiest to see what is happening if one rewrites it as

$$(x-3)^2 + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1.$$

This is an ellipse with center at $(3, 0)$ in the xy plane and going through the points $(2 \pm 1,\, 0)$ and $\left(3,\, 0 \pm \frac{1}{2}\right)$, that is, $(2,\, 0)$, $(4,\, 0)$, $\left(3,\, \frac{1}{2}\right)$, and $\left(3,\, -\frac{1}{2}\right)$.

Here are two versions of the graph, both made with low grade software. In the first one, the y-axis is drawn at $x = 1.5$, rather than the usual $x = 0$.





As it says in the general instructions, axes must be labelled and have scales. $\square$

The equation

$$\frac{(x-3)^2}{4} + y^2 + \frac{12}{16} = 1$$

does define an ellipse, but this ellipse does *not* have semi-axes of lengths 2 and 1. As can be seen from the solution, the semi-axes have lengths 1 and $\frac{1}{2}$.

Problem 9 (12 points)**.** Let $\mathbf{u}$ and $\mathbf{v}$ be functions whose values are three dimensional vectors. Suppose that $\mathbf{u}'(t) = -7\mathbf{u}(t)$ and $\mathbf{v}'(t) = 2\mathbf{v}(t) + \mathbf{i}$. Find $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t))$. (Your answer might involve $\mathbf{u}(t)$ and $\mathbf{v}(t)$, but may not involve $\mathbf{u}'(t)$ and $\mathbf{v}'(t)$.) Be sure to simplify your answer.

Solution. Use the product rule for the cross product and substitute the given information:

$$\begin{aligned} \frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) = (-7\mathbf{u}(t)) \times \mathbf{v}(t) + \mathbf{u}(t) \times (2\mathbf{v}(t) + \mathbf{i}) \\ &= -7\mathbf{u}(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times (2\mathbf{v}(t) + \mathbf{i}) = -5\mathbf{u}(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{i}. \end{aligned}$$

(The simplification is *required.*) $\square$

Remark. An answer using $\mathbf{v}'(t) \times \mathbf{u}(t)$ in place of $\mathbf{u}(t) \times \mathbf{v}'(t)$ is not correct. In fact,

$$\mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{v}'(t) \times \mathbf{u}(t) = \mathbf{u}'(t) \times \mathbf{v}(t) - \mathbf{u}(t) \times \mathbf{v}'(t)$$

(because $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$), while the correct formula gives $\mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$. $\square$

Among other things, it is *required* to simplify $(-7+2)\mathbf{u}(t) \times \mathbf{v}(t)$ to $-5\mathbf{u}(t) \times \mathbf{v}(t)$.

Extra credit. (Do not attempt these problems until you have done and checked your answers to all the ordinary problems on this exam. They will only be counted if you get a grade of B or better on the main part of this exam.)

Problem 10 (15 extra credit points)**.** Define

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^3 + \arctan(x_3 x_4) + \cos(x_1^2 - 2x_1 x_4) + 2x_4.$$

Find an equation for the tangent hyperplane to the level hypersurface of this function which goes through the point $(-2,\ 3,\ 0,\ -1)$.

All work must be shown, using fully correct notation; no credit for just the answer. In particular, intermediate mistakes will lose many points even if they don't lead to an incorrect final answer.

Solution. We calculate the four dimensional gradient:

$$\begin{aligned}
D_1 Q(x_1, x_2, x_3, x_4) &= 2x_1 - \sin(x_1^2 - 2x_1 x_4)(2x_1 - 2x_4), \\
D_2 Q(x_1, x_2, x_3, x_4) &= 3x_2, \\
D_3 Q(x_1, x_2, x_3, x_4) &= \frac{x_4}{1 + (x_3 x_4)^2}, \\
D_4 Q(x_1, x_2, x_3, x_4) &= \frac{x_3}{1 + (x_3 x_4)^2} - \sin(x_1^2 - 2x_1 x_4)(-2x_1) + 2.
\end{aligned}$$

(All these must be done correctly, even though some of the terms vanish at $(-2,\ 3,\ 0,\ -1)$.)

Therefore

$$D_1 Q(-2,\ 3,\ 0,\ -1) = -4, \qquad D_2 Q(-2,\ 3,\ 0,\ -1) = 27,$$
$$D_3 Q(-2,\ 3,\ 0,\ -1) = -1, \qquad \text{and} \qquad D_4 Q(-2,\ 3,\ 0,\ -1) = 2.$$

Thus

$$\mathbf{grad}(Q)(-2,\ 3,\ 0,\ -1) = \big\langle -4,\ 27,\ -1,\ 2 \big\rangle,$$

and an equation for the tangent plane is

$$\big\langle -4,\ 27,\ -1,\ 2 \big\rangle \cdot \big(\langle x_1, x_2, x_3, x_4 \rangle - \langle -2,\ 3,\ 0,\ -1 \rangle \big) = 0,$$

which simplifies to

$$-4x_1 + 27x_2 - x_3 + 2x_4 = 87.91.$$

(Simplification is required.) □

Problem 11 (15 extra credit points)**.** Find a differentiable function f such that

$$D_1 f(x, y) = xy^2 \cos(xy) + 7 \qquad \text{and} \qquad D_2 f(x, y) = x^2 y \cos(xy) + 2y$$

for all x and y, or explain why no such function exists.

All the credit is for the method; no credit for just writing down the answer.

Solution. Integrate:

$$f(x, y) = \int D_1 f(x, y)\, dx = \int (xy^2 \sin(xy) + 7)\, dx.$$

The first term can be integrated by parts, and one gets

$$f(x, y) = xy \sin(xy) + \cos(xy) + 7x + C(y),$$

where the constant of integration $C(y)$ *depends on* y (since we are integrating with respect to x for each fixed y). Now differentiate this formula for f with respect to y, getting

$$D_2 f(x, y) = x^2 y \cos(xy) + C'(y).$$

If $C(y) = y^2$, this agrees with the formula for $D_2 f(x, y)$ given in the problem. So we can take $f(x, y) = xy \sin(xy) + \cos(xy) + 7x + y^2$. □

Any function of the form $f(x, y) = xy\sin(xy) + \cos(xy) + 7x + y^2 + c$, for some constant c, will also work.

The converse (in the relevant sense) of Clairaut's Theorem is false. That is, just because

$$\frac{\partial}{\partial y}\big(xy^2\cos(xy) + 7\big) = \frac{\partial}{\partial x}\big(x^2y\cos(xy) + 2y\big)$$

on some open subset D of $\mathbb{R}^2$, it does not follow that there is a function f such that

$$D_1f(x, y) = xy^2\cos(xy) + 7 \quad \text{and} \quad D_2f(x, y) = x^2y\cos(xy) + 2y$$

for all x and y. (This is true on *certain* open subsets D of $\mathbb{R}^2$, but using this fact requires that you explain why the relevant open set is one of them. In any case, this doesn't help you actually find f.)