Tu 13 Oct. 2020  Math 281

\[ \frac{z}{3} = \frac{x^2}{2^2} + \frac{y^2}{4^2} \quad \frac{z}{3} = (\frac{x}{2})^2 + (\frac{y}{4})^2. \]

2 and 4 are the axes of
both of the two axes of
one of traces (an ellipse),
namely at \( z = 3 \).

At \( z = 3 \):
\[ \frac{x^2}{2^2} + \frac{y^2}{4^2} = 1 \]

The points \((\pm 2, 0, 3)\) and \((0, \pm 4, 3)\) are on the surface.

In the plane \( z = 3 \) we get a trace

\[ \frac{z}{3} = \frac{x^2}{4} - \frac{y^2}{16} = \left(\frac{x}{2}\right)^2 - (\frac{y}{4})^2. \]

Picture: see book. (more is too messy to make sense of)

Vertical traces, in planes parallel to \( xz \) plane or \( yz \) plane
will be parabolas, some facing up, some down.

In boxed eqn: if \( x = 3 \), get
\[ \frac{z}{3} = \frac{9}{4} - \frac{y^2}{16} \quad \text{so} \quad z = \frac{9}{2} - \frac{y^2}{8} \quad \text{parabola facing down.} \]

If take \( y = 4 \), get
\[ \frac{z}{3} = \frac{x^2}{4} - 1 \quad \text{or} \quad z = \frac{x^2}{4} - 3 \quad \text{parabola facing up.} \]
The surface is shaped like a saddle.
Horizontal forces will be impossible.
At \( z = 3 \), get \( 1 = (\frac{x}{2})^2 - (\frac{y}{4})^2 \)
At \( z = -6 \), get \( -2 = \frac{x^2}{4} - \frac{y^2}{16} \)

Now:

In both problems, scales not to
same as x or the axes.

At \( z = 0 \) get \( \frac{x^2}{4} = \frac{y^2}{16} \), the line which crosses \( 4x = \pm y \).
\( y = \pm 4x \).

At \( z = 0 \) is a saddle point
(flatter, looks like a mountain pass).

Core (really, a double cone).
Basic eqn is \( z^2 = x^2 + y^2 \).

Two cones which meet at the origin.
Example Identity and draw \( y^2 = x^2 + 4z^2 + 2x + 5 \).

Complete the square in \( x \): \( y^2 = (x+1)^2 + 4z^2 + 4 \).

Observe that \( -2 < y < 2 \) is not possible: so hyperbolid of two sheets.

The surface is in red.
It opens up in direction of positive
and negative \( z \)-axes.

By mistake, I drew a section of \( y^2 - x^2 + 4z^2 = 4 \).

The correct surface has center at \((-1, 0, 0)\) not at \((0, 0, 0)\).

"Gliders" not necessarily quadratic.
Any surface which depends on only two of the three variables.
Ex. \( x + y^2 = 0 \), or \( x = -y^2 \)

Trees in all horizontal planes are parabolas, always the same:
\[ x = -y^2 \]

"Cylinder" is strange terminology, but is standard (not just here)

Ex. \( z = \sin(y) \)

New independent of \( x \). Trees in planes "\( x = \text{constant} \)" are cross-sections of \( z = \sin(y) \)

Still called a **cylinder**.

The conventional kind of cylinder is called here a "right circular cylinder"; only here only got the infinitely long version.

Sec 13.1 Vector valued functions.

In one variable calculus, have for example a function like
\[ f(x) = \frac{1}{\sqrt{4-x^2}} \] with domain \((-2, 2)\), and values in \( \mathbb{R} \).
\[ f: (-2, 2) \rightarrow \mathbb{R} \]

\[ \text{Domain} \]

Second way to generalize:

\[ r: (-2, 2) \rightarrow \mathbb{R}^2 \text{ or } (-2, 2) \rightarrow \mathbb{R}^3 \]

Values are now vectors (or points).

Consider \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \), a function of three variables.

\[ r(t) = \left\langle \frac{t}{\sqrt{1-t^2}}, e^{-t}, \frac{1}{t^2} \right\rangle \]

This is a function from some subset of \( \mathbb{R} \) to \( \mathbb{R}^3 \), a vector-valued function of one variable.

It might describe the position of a particle at time \( t \).

What is its domain? Need \( t > 0 \) because of the last coordinate.

And need \(-1 < t < 1\) because of first coordinate. So domain is \((0, 1]\) (use both conditions).

Example functions of several variables.

\[ f(x, y, z) = e^{-x} + y^2 + (1 + xz)^4(1 + \cos(yz)) \]

A function \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) has three coordinate functions, each of which is a function of three variables could be the force at point \((x, y, z)\).

Density of smoothen at point \((x, y, z)\).

Vector-valued function of one variable are simpler than functions of several variables.
Let's take for a while \( r(t) = \langle f(t), g(t), h(t) \rangle \).

Almost everything reduces to the behavior of \( f, g, h \), as in one variable calculus.

**Example:**

\[
\lim_{t \to 0} r(t) = \langle \lim_{t \to 0} f(t), \lim_{t \to 0} g(t), \lim_{t \to 0} h(t) \rangle.
\]

Need all three limits in the right to exist. (This is a theorem.)

\[
\lim_{t \to 0} \left[ t^3 \hat{i} - \frac{e^t - 1}{t} \hat{j} + \frac{1}{\cos(2t)} \hat{k} \right].
\]

\[
\lim_{t \to 0} t^2 = 0, \quad \lim_{t \to 0} \frac{e^t - 1}{t} = 1, \quad \text{and} \quad \lim_{t \to 0} \frac{1}{\cos(2t)} = 1.
\]

So known in \(-\hat{j} + \hat{k}\). But \( r \) as above is continuous at \( t = 0 \) if and only if \( f, g, h \) are all continuous at \( t = 0 \).

The fun in previous example is continuous on its domain, which is all real numbers except \( 0 \), \( \pm \frac{\pi}{4} \), \( \pm \frac{\pi}{2} \), \( \pm \frac{3\pi}{4} \), \( \pm \frac{\pi}{4} \), \( \pm \frac{\pi}{2} \), \( \pm \frac{3\pi}{4} \), etc.