W 14 Oct. 2020 Math 281

Vector valued functions of one variable

We looked at continuity and limits.

~~Definition~~ Take for a while $r(t) = \langle f(t), g(t), h(t) \rangle$.

Then $r'(t)$, if it exists, is $\lim_{h \to 0} \frac{1}{h} \left[r(t+h) - r(t) \right]$

Thm r is diff. at t if and only if all of f, g, h are diff. at t,

and then $r'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

no restriction.

Ex Suppose $r(t) = \langle \sqrt{1-t^2}, e^{-t}, \frac{1}{\sqrt{t}} \rangle$.

$-1 \le t \le 1$

$t > 0$.

[note: domain: $(0, 1]$].

To find $r'(t)$, rewrite $r(t) = \langle (1-t^2)^{1/2}, e^{-t}, t^{-1/2} \rangle$.

So $r'(t) = \langle \frac{1}{2}(1-t^2)^{-1/2}(-2t), -e^{-t}, -\frac{1}{2} t^{-3/2} \rangle$.

$= \langle -t(1-t^2)^{-1/2}, -e^{-t}, -\frac{1}{2} t^{-3/2} \rangle$.

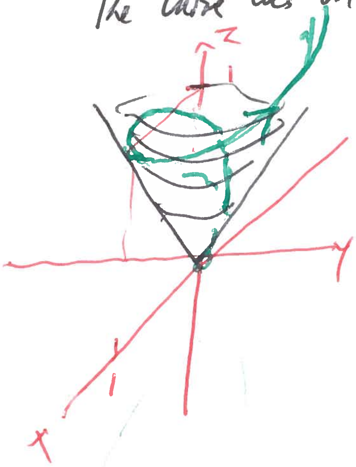
Simplify!

Let's try to draw one example of a parametrized curve in $\mathbb{R}^3$.

$r(t) = \langle t\cos(2\pi t), t\sin(2\pi t), t \rangle$ for $t \ge 0$.

make it easier to draw.

The curve lies on the surface $x^2 + y^2 = z^2$, a double cone, in fact on the top half



At $t = 1$ we are at $\langle 1, 0, 1 \rangle$

At $t = \frac{1}{2}$ $\langle -\frac{1}{2}, 0, \frac{1}{2} \rangle$

$t = \frac{1}{4}$ $\langle 0, \frac{1}{4}, \frac{1}{4} \rangle$

$t = \frac{3}{4}$ $\langle 0, -\frac{3}{4}, \frac{3}{4} \rangle$

Curve in green.

Scales on axes are ~~dif~~ different.

If allow $t<0$, then that part spirals downwards.

Ex of finding a parametrization:

Line segment from $p=(2,3,7)$ to $q=(-1,3,0)$.

Take $r(t) = \langle 2,3,7\rangle + t\left(\langle -1,3,0\rangle - \langle 2,3,7\rangle\right)$

$= \langle 2,3,7\rangle + t\langle -3,0,-7\rangle = \langle 2-3t,\ 3,\ 7-7t\rangle$.

Back to derivatives.

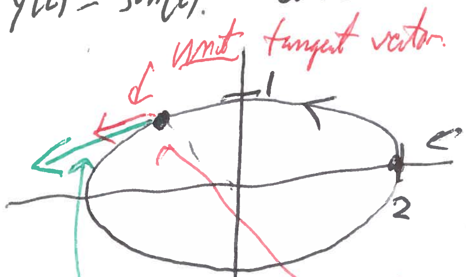
Consider $r(t) = \langle f(t), g(t), h(t)\rangle$.

Then $r'(t)$ is the rate of change of $r(t)$.

If $r(t)$ is the ~~position~~ position of a particle at time t, then $r'(t)$ is its velocity at time t.

It has both direction and magnitude, here: direction of motion, speed (how fast it is going).

Ex in $\mathbb{R}^2$, so can draw it. Take $r(t)=\langle x(t), y(t)\rangle$ with $x(t) = 2\cos(t)$, $y(t)=\sin(t)$. Curve is on the ellipse $\left(\frac{x}{2}\right)^2 + y^2 = 1$.



$\leftarrow$ starts here at time 0.

Look at $t = \frac{3\pi}{4}$.

$r'(t) = \langle -2\sin(t), \cos(t)\rangle$

$r'\left(\frac{3\pi}{4}\right) = \left\langle -\sqrt{2}, -\frac{\sqrt{2}}{2}\right\rangle$

$r\left(\frac{3\pi}{4}\right) = \left\langle -\sqrt{2}, \frac{\sqrt{2}}{2}\right\rangle$

vector $r'\left(\frac{3\pi}{4}\right)$ shown starting at $r\left(\frac{3\pi}{4}\right)$.

Observe: $r'(\frac{3\pi}{4})$ is tangent to the curve at the point $r(\frac{3\pi}{4})$, and points in the direction of motion.

There is a speed, $\|r'(t)\| = \sqrt{(-\sqrt{2})^2 + (-\frac{\sqrt{2}}{2})^2} = \sqrt{\frac{5}{2}}$.

There is a direction given by a unit tangent vector, pointing in direction of motion, given by

$$T(t) = \frac{1}{\|r'(t)\|} \cdot r'(t) = \cdots = \langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle.$$

Ex Find a parametrization for the tangent line to the curve at the point $r(\frac{3\pi}{4})$.

Call it $L(t)$. Need a point and a direction.

$r(\frac{3\pi}{4}) = \langle -\sqrt{2}, \frac{\sqrt{2}}{2} \rangle$. $r'(\frac{3\pi}{4}) = \langle -\sqrt{2}, -\frac{\sqrt{2}}{2} \rangle$.
(Could use $T(\frac{3\pi}{4})$ instead)

$$L(t) = \langle -\sqrt{2}, \frac{\sqrt{2}}{2} \rangle + t\underbrace{\langle -\sqrt{2}, -\frac{\sqrt{2}}{2} \rangle}_{\text{Need } r'(\frac{3\pi}{4}), \text{ not } \cancel{r'(t)}}$$

Differentiation rules.

Rules for sums, differences, constant multiples are the same. (Check by coordinates). [see page 848 of book].

Ex of one: $\frac{d}{dt}(u(t)+v(t)) = u'(t) + v'(t)$.

There are three kinds of products:

$\frac{d}{dt}(f(t)\,u(t))$ $\qquad$ f scalar valued, u vector valued.

$\frac{d}{dt}\left(u(t)\cdot v(t)\right)$, $\frac{d}{dt}\left(u(t)\times v(t)\right)$.

They all follow an analog of the product rule.

Ex: $\frac{d}{dt}\left(u(t)\times v(t)\right) = u'(t)\times v(t) + u(t)\times v'(t)$

[Note: keep the order! since $\times$ is not commutative]

Why? Only look at first coord.

$u(t)\times v(t) = \langle u_2(t)v_3(t) - u_3(t)v_2(t), \text{— — —} \rangle$

Differentiate the coords, using ordinary product rule:

$$\frac{d}{dt}\left(u(t)\times v(t)\right) = \langle u_2'(t)v_3(t) + u_2(t)v_3'(t) - u_3'(t)v_2(t) - u_3(t)v_2'(t), \text{— —} \rangle$$

These are the first coord of $u'(t)\times v(t)$

These are the first coordinate of ~~u(t)~~
$u(t)\times v'(t)$