F 16 Oct. 2020 Math 281.

Diff. rules for vector valued functions

Let u and v be vector valued functions. (For cross product, need to assume in $\mathbb{R}^3$.)

Assume $u(t)=\langle u_1(t), u_2(t), u_3(t)\rangle$
$v(t)=\langle v_1(t), v_2(t), v_3(t)\rangle$

Let h be a scalar valued function.

Let c be a constant,

Assuming all derivatives on the right exist, :

$$\frac{d}{dt}(u(t)+v(t)) = u'(t)+v'(t)$$

$$\frac{d}{dt}(u(t)+v(t)) = \frac{d}{dt}\left(\langle u_1(t)+v_1(t),\ u_2(t)+v_2(t), u_3(t)+v_3(t)\rangle\right).$$

$$= \left\langle \frac{d}{dt}(u_1(t)+v_1(t)), \frac{d}{dt}(\quad), \frac{d}{dt}(\quad)\right\rangle$$

$$= \langle u_1'(t)+v_1'(t),\ u_2'(t)+v_2'(t), \text{---}\rangle$$

$$= u'(t)+v'(t).$$

Need this if one or both of $u(t), v(t)$ is not explicitly given.

$$\frac{d}{dt}(c\,u(t)) = c\,u'(t)$$

$$\frac{d}{dt}(u(t)-v(t)) = u'(t)-v'(t).$$

Chain rule: if g is a function from an $\underbrace{\text{open interval in } \mathbb{R}}_{\text{scalars (parameter values)}}$ to $\mathbb{R}$. ← scalars, (parameter values)

If domains match properly, it makes sense to consider, e.g.

$$r(t) = u(g(t)) = \langle u_1(g(t)), u_2(g(t)), u_3(g(t))\rangle.$$

Then $r'(t) = \langle u_1'(g(t))\,g'(t),\ u_2'(g(t))\,g'(t),\ u_3'(g(t))\,g'(t)\rangle$
$= g'(t)\,u'(g(t))$ (since scalar mult. goes on the left).

That is,

$$\frac{d}{dt}\left(u(g(t)) \right) = g'(t)\, u'(g(t)).$$

$\boxed{\text{Written HW 3 \underline{does} apply to final grade.}}$

Product rules.

$$\frac{d}{dt}\left(h(t)\, u(t) \right) = \underbrace{h'(t)}_{\text{scalar}}\, \underbrace{u(t)}_{\text{vector}} + \underbrace{h(t)}_{\text{scalar}}\, \underbrace{u'(t)}_{\text{vector}}.$$

$$\frac{d}{dt}\left(u(t) \cdot v(t) \right) = \underbrace{\underbrace{u'(t)}_{\text{vector}} \cdot \underbrace{v(t)}_{\text{vector}}}_{\text{scalar}} + u(t) \cdot v'(t).$$

same here.

$$\frac{d}{dt}\left(u(t) \times v(t) \right) = \underbrace{u'(t) \times v(t)}_{} + \underbrace{u(t) \times v'(t)}_{}$$

For part of this, see last time (end).

The order matters!

[It works because all have distributive laws.]

Recall physics interpretation: If $v(t)$ is position of a particle at time t, then $v'(t)$ is its velocity.

[When one does need to distinguish vectors and points, such as replacing $\mathbb{R}^3$ with a curved surface, then position is a point and velocity is a vector.]

Suppose we are given velocity $v(t)$ of a particle at time t, and say position at time t_0. Want to find position at time t.

In one dimension: Form $\displaystyle\int_{t_0}^{t} v(t)\,dt$ (and add position at time t_0).

Same in 2, 3, or many dimensions.

Definition of integral of vector valued function is similar.

Thm If $r(t) = \langle f(t), g(t), h(t) \rangle$, then $\int_a^b r(t)dt = \langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \rangle$

Also: Write antideriv. as $\int r(t)dt$. It is $\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \rangle$.

Ex Find indef. integral of $r(t) = \langle t^2, t^3, t^6 \rangle$.

[The ~~first~~ Fundamental Theorem of Calculus works the same way as for one variable.]

Answer: $\int r(t)dt = \langle \frac{1}{3}t^3 + C_1, \frac{1}{4}t^4 + C_2, \frac{1}{7}t^7 + C_3 \rangle$

$= \langle \frac{1}{3}t^3, \frac{1}{4}t^4, \frac{1}{7}t^7 \rangle + C$

vector constant, $C = \langle C_1, C_2, C_3 \rangle$.

C_1, C_2, C_3 are scalar constants.

Ex $\int_2^3 (\cos(t)\, i - \sin(2t)\, j - e^{-t} k)dt$.

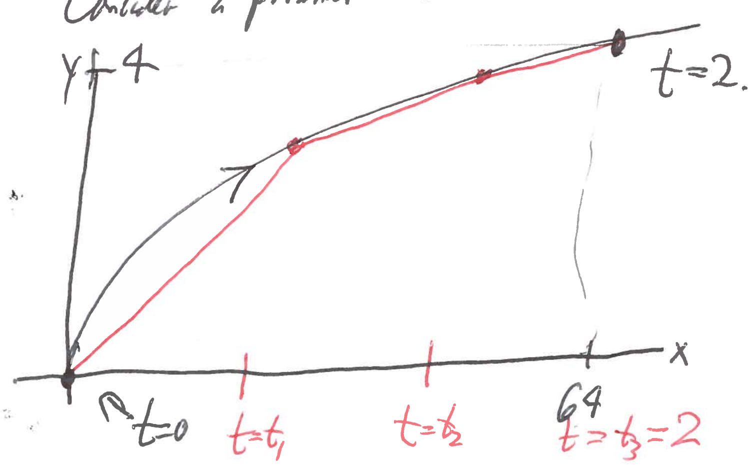
parentheses needed.

$= \left(\sin(t)\Big|_2^3\right) i + \left(\frac{1}{2}\cos(2t)\Big|_2^3\right) j + \left(e^{-t}\Big|_2^3\right) k$.

$= [\sin(3) - \sin(2)]\, i + \text{— — —}$

Arc length. Start in $\mathbb{R}^2$, where I can draw it. Works in $\mathbb{R}^n$ for any n.

Consider a parametrized curve: $x(t) = t^6$, $y(t) = t^2$.



It lies on the graph of $x = y^3$ or $y = \sqrt[3]{x}$, for $x \geq 0$.

Consider length from $t=0$ to $t=2$

Assume $x'(t)$, $y'(t)$ continuous

$t_0 = 0,\ t_1,\ t_2,\ t_3 = 2.$

Points are $(x(t_0), y(t_0))$, $(x(t_1), y(t_1))$, $(x(t_2), y(t_2))$, $(x(t_3), y(t_3))$.

Total length of polygonal approximation is

$$\sqrt{[x(t_1)-x(t_0)]^2 + [y(t_1)-y(t_0)]^2} + \text{(two more similar terms)}$$

$$= \left[\sqrt{\left(\frac{x(t_1)-x(t_0)}{t_1-t_0}\right)^2 + \left(\frac{y(t_1)-y(t_0)}{t_1-t_0}\right)^2}\right](t_1-t_0) + \text{(two more similar terms)}.$$

$$\approx \sqrt{(x'(t_0))^2 + [y'(t_0)]^2}\ \Delta t_1 + \text{(two similar terms)}$$

if t_1 close enough to t_0

When mesh of subdivision goes to 0, get $\int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$