Recall: A function $f$ with domain in $\mathbb{R}^2$ (or $\mathbb{R}^n$ with change of notation) is continuous at $(x,y)$ if

$$f(a,b) \text{ is defined, and lim } f(x,y) = f(a,b).$$

$f$ is continuous on a set $D$ in $\mathbb{R}^2$ (or $\mathbb{R}^n$) if $f$ is cont. at each point of $D$.

**Exs:**

1. $f(x,y,z) = 2$ for all $x,y,z$ real (all pts. $(x,y,z)$ in $\mathbb{R}^3$). Cont. on $\mathbb{R}^3$.

2. $g(x,y) = x^2$ for all $x,y$ real (all pts. $(x,y)$ in $\mathbb{R}^2$) Cont. on $\mathbb{R}^2$.

3. $k(x,y,z) = x^2 - y^2$ Cont. on $\mathbb{R}^3$.

General principle: Except in very peculiar cases, functions given by formulas will be cont on domain.

We will use continuity + limits of several variables slightly differently.

We will define $f_x(x,y)$ (or $\frac{\partial f}{\partial x}$ at $(x,y)$) by fixing $y$, considering the one variable function $x \mapsto f(x,y)$, and differentiating with respect to $x$. Similarly for others. This is not really new, but uses one variable limits.

It is possible for $f_x(0,0)$ and $f_y(0,0)$ to exist, but for $f$ to be not even continuous at $(0,0)$! We rule out this case & its consequences by requiring the partial derivatives to be cont.

**Upshot (also for other reasons):** We need to pay more attention to continuity than in Math 25.

More exs:

1. $f(x,y) = \left\{ \begin{array}{ll} \frac{xy}{x^2 + y^2} & (x,y) \neq (0,0) \\ \text{undefined} & (x,y) = (0,0) \end{array} \right.$

   This fun is cont. on its domain.

As in one variable, sums, differences, scalar multiples, products of cont. funs are cont.

$$x \mapsto \frac{f(x,y)}{g(x,y)} \quad \text{for } f, g \text{ cont } \implies x \mapsto \text{cont., where the denominator is not zero}$$

Also, if $h$ is a cont. fun of one variable, and $f$ is a cont. fun on $D$ subset of $\mathbb{R}^3$ (or $\mathbb{R}^n$), then $(x,y) \mapsto h(f(x,y))$ is cont. in $D$ (if $h$ defined on range of $f$)

**Exs:**

1. $g(x,y) = \sin \left( \frac{xy}{x^2 + y^2} \right)$ is continuous on $\mathbb{R}^2$ except $(0,0)$,

2. $h(x,y) = \ln \left( \frac{x^4 + y^4}{x^2 + y^2} \right) e^{-y^2}$ is continuous on its domain, which is the set of all pairs $(x,y)$ such that $|x| > |y|$.

   (Exactly when $x^2 > y^2$)

(These dotted lines form the boundary, and are not included.)

Every polynomial of several variables is cont. everywhere. Ex: $p(x,y) = 2x^2 + 3y^2 + 29x y - 25 + 8$
Ex of using derivative. Consider a straight road on hilly terrain.

height (elevation) at horizontal position x is \( h(x) \). Picture shows \( z = h(x) \).

\( h'(2) \) is the slope of road at position \( x = 2 \) in ft/mile. If you go 1 mile/hour horizontally (road speed will be a bit higher), then gaining \( h'(2) \) ft/hour as you go through \( x = 2 \).

Consider now moving over hilly terrain in two dimensions. Draw contour (level curves).

**Consider \( 2,3 \).**

**Suppose we go east at one mile/hour.**

- **Here is elevation changing?**
- **It is increasing, and we can estimate how fast:** a bit less than 100 ft/hour. Maybe about 70?

**Suppose we go north instead.** We are now going down, at about 70 ft/hour.

For going E: partial derivative in the x direction at \( x = 2 \) and \( y = 3 \) is about 70.

- It is \( \frac{\partial}{\partial x} (h(x, 3)) \) here then at \( x = 2 \).

For going N, partial derivative in the y direction at \( x = 2 \) and \( y = 3 \) is about -70.

- It is \( \frac{\partial}{\partial y} (h(2, y)) \) evaluated at \( y = 3 \).

**Notation:** \( h_x (x, y) \) (here \( h_x (2, 3) \)) and \( h_y (x, y) \)

**Suppose we start at \( (2,3) \) and move with velocity vector \( \langle 1, 1 \rangle \).** (going NE at \( \sqrt{2} \) mile/hour). If the function is reasonable, then velocity change will be

\( f_x (2,3) \cdot 1 + f_y (2,3) \cdot 1 \approx 0 \) in the picture. (black arrow)

**Ex:** velocity \( \langle 4,5 \rangle \) (now going roughly NE, a bit more north than east at \( y \) ft/mile/hour). Rate of change of elevation will be \( f_x (2,3) \cdot 4 + f_y (2,3) \cdot 5 \)

- here \( \approx (70)(4) + (-70)(3) = -70 \).

This is an example of the chain rule in several variables. Not enough for the partial derivatives to just exist, enough to require \( f_y, f_z \) to be continuous.
Suppose \( f(x, y) = x + 2y^2 + 3x^2 \).

Let's find \( f_x(4, 5) \). Then \( \frac{df}{dx} f(x, 5) \), evaluated at \( x = 4 \).

\[
f(x, 5) = x + 2(5)^2 + 3x^2.5 = x + 50 + 15x^2
\]

Deriv. \( 1 + 30x \), so \( f_x(4, 5) = 1 + 30(4) = 121 \).

What is \( f_y(x, y) \)?