Thus is what we had last time:

Suppose $f(x,y) = x + 2y^2 + 3xy^2$. 

Let's find $f_x(4,5)$.

Thus $\frac{d}{dx} f(x,y)$, evaluated at $x=4$.

\[
f(x,y) = x + 2(5)^2 + 3x^2 \cdot 5 = x + 50 + 15x^2
\]

Deriv: $1 + 30x$, so $f_x(4,5) = 1 + 30(4) = 121$.

What is $f_y(x,y)$?

Let's first do $f_y(4,5)$ the same way. First find $f_y(4,y) = 4 + 2y^2 + 3(42)y$.

So $f_y(4,y) = 4y + 48$, and $f_y(4,5) = 20 + 48 = 68$. 

\[\text{Note: not specifically } y=5, \text{ but pretend } y \text{ is a constant.}
\]

Now $f_x(x,y) = \frac{\partial}{\partial x} \left( x + 2y^2 + 3x^2y \right) = 1 + 0 + 6xy = 1 + 6xy$.

Cautions: Do not confuse with implicit differentiation in Math 25.

\[f_y(x,y) = \frac{\partial}{\partial y} \left( x + 2y^2 + 3x^2y \right) = 0 + 4y + 3x^2 = 4y + 3x^2.
\]

Here, $x$ is treated as a constant.

\[F_x = g(x,y,z) = x \cos(y^2 + yz) \]

\[F_y = g_x, g_y, g_z \quad \text{with product rule.}
\]

\[g_y(x,y,z) = \frac{\partial}{\partial y} (x \cos(y^2 + yz)) = x \frac{\partial}{\partial y} (\cos(y^2 + yz))
\]

\[\text{Use constant multiple rule.}
\]

\[= x \left[ -\sin(y^2 + yz) \right] \frac{\partial}{\partial y} (y^2 + yz) = -x \sin(y^2 + yz)(2y + yz)
\]

\[g_z(x,y,z) = \frac{\partial}{\partial z} (x \cos(y^2 + yz)) = x \left[ -\sin(y^2 + yz) \right] \frac{\partial}{\partial z} (y^2 + yz)
\]

\[\text{Use constant multiple rule.}
\]

\[= -x \sin(y^2 + yz)(3yz^2) = -3x \sin(y^2 + yz)(yz^2),
\]

\[\text{since } x \text{ is taken to be constant.}
\]
Interpretation (again). Rate of change at a trace. Take \( f(x, y) = \sqrt{21-x^2-y^2} \)

Graph is a hemisphere:

Consider traces of this surface in the plane \( y = 1 \) and the plane \( x = 2 \).

Get eqn \( f(x, -1) = \sqrt{21-x^2-(x)^2} \), \( f(2, y) = \sqrt{21-2^2-y^2} \).

Expect \( f_x(2, -1) \) to be negative. Expect \( f_y(2, -1) \) to be positive.

\[ f(x, y) = \sqrt{21-x^2-y^2} \]
\[ f_x(x, y) = \frac{1}{2} (21-x^2-y^2)^{-1/2} \cdot \frac{\partial}{\partial x} (21-x^2-y^2) = \frac{1}{2} \left( -\frac{x}{21-x^2-y^2} \right) \]

and \( f_x(2, -1) = \frac{-2}{\sqrt{21-2^2-(-1)^2}} = \frac{-2}{\sqrt{16}} = -\frac{1}{2} \).

Also \( f_y(x, y) = \frac{1}{2} (21-x^2-y^2)^{-1/2} (-2y) = -\frac{y}{\sqrt{21-x^2-y^2}} \).

\( f_y(2, -1) = \frac{-1}{2} \).

Go back \( f(x, y) = x+2y^2+3x^2 \).

We had \( f_x(x, y) = 1+6xy, f_y(x, y) = 4y+3x^2 \).

What is \( (f_x)_x (x, y) \)? \( \frac{\partial^2}{\partial x^2} (f_x(x, y)) = \frac{\partial}{\partial x} (1+6xy) = 6y \).

Abbreviate \( f_{xx}(x, y) \).

\( (f_y)_y (x, y) = \frac{\partial^2}{\partial y^2} (4y+3x^2) = 4 \).

Abbreviate \( f_{yy}(x, y) \).

\( f_{xy} (x, y) = (f_x)_y (x, y) = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f(x, y) \right) = \frac{\partial}{\partial y} (1+6xy) = 6x \).

\( f_{yx} (x, y) = (f_y)_x (x, y) = \frac{\partial}{\partial x} (4y+3x^2) = 6x \).
Theorem: If \( f_{xy} \) and \( f_{yx} \) are both continuous, then they are equal.

(by Cauchy's theorem, this can be false.)

Other notation: \( \frac{\partial^2}{\partial y \partial x} \) for \( f_x \), \( \frac{\partial^2}{\partial x \partial y} \) for \( f_y \)

Similar theorems for higher order: If they are all continuous then \( f_{xy}, f_{yx}, f_{xx}, \) and \( f_{yy} \) are all equal.

Partial differential equations:

Example: One thin wave eqn: \( \frac{\partial^2 u}{\partial t^2} (x,t) = c \frac{\partial^2 u}{\partial x^2} (x,t) \) for some constant \( c > 0 \)

\( u(x,t) \) is height of wave at position \( x \) and time \( t \).

\( T(t,x,y,z) \) temp at time \( t \), position \( (x,y,z) \), in metal rod.