If \( f \) is a function of two variables and \( f \) is differentiable at \((x_0, y_0)\), with \( f(x, y) = z_0 \), then the equation of the tangent plane to the graph of \( f \) at \((x_0, y_0)\) is 
\[ z = z_0 = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0). \]
(Similarly for more variables.)

By definition (not stated formally), \( f \) is differentiable at \((x_0, y_0)\). Strictly speaking, this plane really is tangent to the graph in a suitable sense. Tangent plane is the graph of the linear approx. to \( f \) at \((x_0, y_0)\).

Example: Suppose \( f \) itself is unknown, but you know \( f \) is differentiable at \((2, -1, 1)\), \((2, -1, 1)\) (if \( f \) is a function of three variables), and you know \( f_x(2, -1, 1) = 3, f_y(2, -1, 1) = 4, f_z(2, -1, 1) = -2. \) Approximate \( f(1.98, -1.03, 1.1) \)

\[
L(x, y, z) = f(2, -1, 1) + 3(x-2) + 4(y+1) - 2(z-1)
\]

Linear approx. to \( f \)

\[
f(1.98, -1.03, 1.1) \approx L(1.98, -1.03, 1.1) = 6 + 3(1.98 - 2) + 4(-1.03 + 1) - 2(1.1 - 1)
\]

\[
= 5.62.
\]

Recall: If \( f_x \) and \( f_y \) are defined near \((x_0, y_0)\) and continuous there, then \( f \) is differentiable at \((x_0, y_0)\).

Not enough for \( f_x, f_y \) to just exist.

Example: \( f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \)

We already saw that \( f \) is not contin. at 0, since

\[
\lim_{t \to 0^+} f(0, t) = \frac{1}{2} \neq f(0, 0).
\]

Look at \( f_x(0, 0) \).

Consider the function \( x \mapsto f(x, 0) \) and differentiate at \( x = 0 \).

\[
f(x, 0) = \begin{cases} \frac{x^2}{x^2 + 0^2} & x \neq 0 \\ 0 & x = 0 \end{cases}
\]

So \( f_x(x, 0) = 0 \) for all \( x \).

Therefore \( f_x(x, 0) = 0 \) and \( f_y(0, 0) = 0 \).

Similarly, \( f(0, y) = 0 \) for all \( y \), so \( f_y(0, 0) = 0 \).

If there were, \( \lim \text{approx. at } (0, 0) \) would be \( L(x, y) = 0 \) for all \( x, y \).

Hence, \( f(0.01, 0.01) = \frac{1}{2} \) not close to 0, etc. \( f \) is not even continuous at \((0, 0)\).

Can check \( f_x(0, y) \) for \((x, y) \neq (0, 0) \) in (and near) \( y^2 - x^2 ) \)

At \((0, 0)\) for \( t \to 0 \), get \( \frac{1}{t} \) which \( \to \infty \) as \( t \to 0. \) \( f_x \) is not contin. at \((0, 0)\).
Infrad def: \( f \) is diff at \((a,b)\) if the supposed tangent plane approximation really is.

First order approximation:
\[
| f(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0) | \leq h(\Delta x, \Delta y) \sqrt{\Delta x^2 + \Delta y^2}
\]

with \( h(\Delta x, \Delta y) \rightarrow 0 \) as \((\Delta x, \Delta y) \rightarrow (0,0)\)

Thus, if \( f_x, f_y \) are cont. near \((a,b)\), then \( f \) is diff at \((a,b)\).

(Converse is false)

Ex of tangent plane: \( f(x,y) = \ln(2x-y) + x^2 - y^2 \) at \((2,3)\).

We need \( f(2,3) = \ln(1) + 2^2 - 3^2 = -3 \).

\[
f_x(x,y) = \frac{2}{2x-y} + 2x \quad \text{and} \quad f_y(x,y) = \frac{-1}{2x-y} - 3y^2
\]

These are continuous near \((2,3)\), so \( f \) really is diff at \((2,3)\).

\[
f_x(2,3) = 2 + 2 \cdot 2 = 6 \quad \text{and} \quad f_y(2,3) = -1 - 3 \cdot 3^2 = -28.
\]

So, tangent plane has eqn \( z = -23 = 6(x-2) - 28(y-3) \). Simplify... [omitted]

Multivariable chain rule.

Keep notation hom: consider explicit case. \( z(t) = f(g(t), h(t)) \), \( f \) is \( \geq \) on \( t \).

We have \( z(t) \) as functions of one variable. Implem. \( z = f(x(t), x = g(t), y = h(t)) \).

What is \( z'(t) \)? To make it look more like 1 variable chain rule, set \( r(t) = (g(t), h(t)) \).

Then \( z(t) = f(r(t)) \).

Consider linear approximation to \( f \) at \((x,y) = (g(t), h(t)) \) (here \( t \) is fixed).

\[
f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0) + f_x(x_0) \Delta x + f_y(x_0) \Delta y
\]

One variable lin approx.
\[
g(t + \Delta t) \approx g(t) + g'(t) \Delta t \quad \text{and} \quad h(t + \Delta t) \approx h(t) + h'(t) \Delta t
\]

Thus \( z(t + \Delta t) \approx f(g(t + \Delta t), h(t + \Delta t)) \approx f(g(t), h(t)) + f_x(g(t), h(t)) g'(t) \Delta t + f_y(g(t), h(t)) h'(t) \Delta t \)

Since this is the lin approx to \( z(t + \Delta t) \), must be \( z'(t) \).

So \( z'(t) = f_x(g(t), h(t)) g'(t) + f_y(g(t), h(t)) h'(t) \).

In physics, notation:
\[
\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}
\]