Chain rule: Simplest case: \(z(t) = f(g(t), h(t))\). Then \(z'(t) = f'(g(t), h(t))g'(t) + f'(g(t), h(t))h'(t)\).

If \(x = g(t), y = h(t)\), in physicists' notation, we write \(\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\).

Ex. Suppose \(f(x, y) = x^2 - \sin(x^2 + xy)\) and \(x(t) = t^2, y(t) = t^3\). We have \(z(t) = f(x(t), y(t))\).

For \(z(t) = f(x(t), y(t))\), [Note: Could evaluate \(z(t)\) and differentiate that.]

Solution: \(z'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)\).

Now, \(f_x(x, y) = y^2 - \cos(x^2 + xy)(2x)\) and \(f_y(x, y) = x^2y - \cos(x^2 + xy)\).\(\vdots\)

So \(z'(t) = \left[2t^3 - \cos(t^6 + t^3)^2 b(t) + \left[3t^2(t^3)^2(3t^6b^3 + t^2b^3)\right] b(t)\right];[\ldots] = \text{essential}\).

Then a fair amount of simplification, omit.

Ex. Suppose \(g(x, y) = e^{-x^2 + 2x\sin(x^2 + y)}\) and \(h(x, y) = \sin(x^2 + y)\).

Find deriv. of \(u(x, y) = g(t^2, e^t)\).\n
Solution: \(u'(t) = \left[e^{-t^2 - e^{2t}}(2t + \sin((t^2 + e^t) + \sin((t^2 + e^t)) e^t + 2te^{-t^2 + 4t^3 + e^t} + e^t \sin(t^4 + e^t))\right] - 2te^{-t^2 + 4t^3 + e^t} + e^t \sin(t^4 + e^t)\).

Ex. We are given \(h_x(2, 7) = -3, h_y(2, 7) = 4, f(3) = 2, f'(3) = 6, f(6) = 7, f'(6) = -8\).

\(g(3) = 7, g'(3) = -2, g(6) = 7, g'(6) = -1\).

What derivative can we get from the chain rule? \(z'(3) = f(3) = h(2, 7) = 7\).

0: \(z'(3) = h_x(4, 3)g'(3) + h_y(4, 3)g(3) = h_x(2, 7) + h_y(2, 7)g'(3)\).

\(-3\cdot 6 + 4 (-2) = -26\). Can get \(-26\) since \(f(6) = 7, g(6) = 7\).

Find \(h(7, 7) = h(2, 7)\).

Ex. We suggest \(u(x, y, z) = x^2y^3z^2, u_x(x, y, z) = 2x y^3 z^2, u_y(x, y, z) = 3x^2 y^4 z^2, u_z(x, y, z) = 2x^2 y^3 z^2\).

Find \(\frac{4}{t^2} \left[u(-t^2 - e^4, t^3)\right].\)

\(t^3 = \left[2t^3(-t^4)^2(b^3)\right] (-t^3) + \left[\arctan(t) + 2(-t^3)^2(-t^3)^2\right]^2 \left[ -4t^3\right]

\(+ \left[\sin((t^2)^7) + 4(-t^3)\right] \left(-t^3\right)^3 \left(-t^3\right)^3\] \left[2t^3\right].\)

There is a lot of simplification.
\[ z'(t) = u'(f_1(t), \ldots, f_n(t)) \cdot f'(t) + (D_n u)(f_1(t), \ldots, f_n(t)) f_n'(t). \]

Chain rule for partial derivatives of \( r \) is already something else.

\[ p(r, s) = u(f(r, s), g(r, s)). \]

\[ p_r(r, s) = u_r(f(r, s), g(r, s)) f_r(r, s) + u_s(f(r, s), g(r, s)) g_r(r, s). \]

Or: \( D_r p(r, s) = D_r u(f(r, s), g(r, s)) D_r f(r, s) + D_s u(f(r, s), g(r, s)) D_g g(r, s). \)

(Noting partial derivs by position.)

With more systematic notation:

\[ p(x_1, x_2) = u(f_1(x_1, x_2), f_2(x_1, x_2)). \]

\[ D_1 p(x_1, x_2) = D_1 u(f_1(x_1, x_2), f_2(x_1, x_2)) D_1 f_1(x_1, x_2) + D_2 u(f_1(x_1, x_2), f_2(x_1, x_2)) D_2 f_2(x_1, x_2). \]

\[ D_2 p(x_1, x_2) = D_1 u(f_1(x_1, x_2), f_2(x_1, x_2)) D_2 f_1(x_1, x_2) + D_2 u(f_1(x_1, x_2), f_2(x_1, x_2)) D_2 f_2(x_1, x_2). \]