Correct mathematical way to write partial derivative \( \frac{\partial}{\partial x} f(x, y, z) \), \( \frac{\partial^2}{\partial x^2} f(x, y, z) \): what do \( f_x(y, x) \) mean? What about \( f_y(x, x) \)? I have used the following: \( f \) is a function of the variables, \( z \) is the partial derivative with respect to first variable, and evaluate "on the diagonal" at \( (x, x) \) for \( \text{red } x \).

Chain rule: Suppose \( u = u(x, y) \) is a function of two variables and \( h = h(x, y) \) is a function from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). Then \( u(h(x, y)) \) make sense, and chain rule for this:

\[
\begin{align*}
\frac{\partial}{\partial x} z(x, y) &= D_x u(h(x, y)) D_1 h(x, y) + D_x u(h_1(x, y), h_2(x, y)) D_1 h_1(x, y) + D_x u(h_1(x, y), h_2(x, y)) D_1 h_2(x, y) \\
\frac{\partial}{\partial y} z(x, y) &= D_y u(h(x, y)) D_2 h(x, y) + D_y u(h_1(x, y), h_2(x, y)) D_2 h_1(x, y) + D_y u(h_1(x, y), h_2(x, y)) D_2 h_2(x, y).
\end{align*}
\]

[Should have used \((x, x)\) not \((x, y)\).]

Use Chain rule.

**Example:** Suppose \( f(x, y) = -\sin(x^2 + y^2) \), \( x(s, t) = st^2 \), \( y(s, t) = s + t^2 \). Suppose \( z(x, y) = f(x(s, t), y(s, t)). \)

Then \( f_x(x, y) = D_1 f(x, y) = -\cos(x^2 + y^2)(2x) \) and \( f_y(x, y) = D_2 f(x, y) = -\cos(x^2 + y^2)(2y) \).

Also need: \( D_1 x(s, t) = x_t(s, t) = t^2 \) and three more, which I will put in as needed.

Then \( D_1 z(s, t) = D_1 f(x(s, t), y(s, t)) \cdot D_1 x(s, t) + D_2 f(x(s, t), y(s, t)) D_1 y(s, t) \)

\[
= -\cos((st^2)^2 + (s + t^2)^2)(2st^2 + 2s) t^2
\]

\[
= -\cos((st^2)^2 + (s + t^2)^2) st^2 (1) \cdot D_1 y(s, t).
\]

**Example:** Suppose \( D_1 g(x, y) = e^{-x^2} + 2x \sin(x^2 y) \) and \( D_2 g(x, y) = \sin(x^2 y) \).

Find partial derivatives of \( w(s, t) = g(st^2, st^3) \).

\[
\begin{align*}
D_1 w(s, t) &= D_1 w(s, t) = D_1 g(st^2, st^3) \frac{\partial}{\partial t} (st^2) + D_2 g(st^2, st^3) \frac{\partial}{\partial t} (st^3) \\
&= -e^{-(st^2)^2} \cdot 2st^2 \cdot \sin((st^2)^2 + (st^3)^2) + 2st + \sin((st^2)^2 + (st^3)^2) \cdot 3st^2.
\end{align*}
\]

**Implicit differentiation.**

**Example:** Suppose \( y \) as a function of \( x \) is supposed to be implicitly given by the equation \( x^2 + y^3 + 7x y^5 = 1 \). What \( \frac{dy}{dx} \)?

**CAUTION:** In this context, \( y \) is a function of \( x \), and \( \frac{dy}{dx} \) is not the partial derivative which would be zero. Rather, \( y \) is really \( y(x) \) and \( \frac{dy}{dx} = y'(x) \).
Make things more explicit: \( y(x) \) is a function of \( x \) such that
\[
x^2 + y(x)^2 + 7x y(x)^5 = 1
\]
for all \( x \) in its domain. Recall: Differentiate both sides with respect to \( x \), getting:
\[
2x + 3y(x)^2 \frac{d}{dx} y(x) + 7y(x)^5 + 7y(x)^4 \frac{d}{dx} y(x) = 0.
\]
Solve for \( y'(x) \). I get
\[
y'(x) = \frac{-2x - 7y(x)^5}{2y(x)^2 + 35y(x)^4}.
\]

Rewrite: \( F(x, y(x)) = 0 \) with \( F \) a function of two variables, \( F(x, y) = x^2 + y^5 + 7xy^8 - 1 \).

Then use chain rule on this:
\[
0 = \frac{d}{dx} \left( F(x, y(x)) \right) = D_1 F(x, y(x)) \frac{d}{dx} (x) + D_2 F(x, y(x)) \frac{d}{dx} (y(x))
\]
\[
\frac{d}{dx} (0) = D_1 F(x, y(x)) + D_2 F(x, y(x)) y'(x)
\]
Solve: \( y'(x) = -\frac{D_1 F(x, y(x))}{D_2 F(x, y(x))} \). This will give some answer as before.

\[\text{Ex.}\] \( xyz - z^2 + \sin(y) = 0 \) is supposed to define \( z \) implicitly as a function of \( x, y \).

To reduce chance of mistakes, write it as \( xyz(x, y) - z(x, y)^2 + \sin(y) = 0 \).

\( z \) has two partial derivatives.

For \( D_1 \) (to find \( z_x(x, y) \)): Differentiate eqn with respect to \( x \).
\[
yz(x, y) + xy D_1 z(x, y) + 2z(x, y) D_z z(x, y) + 0 = 0 \quad \text{(Now solve for } D_1 z(x, y)\).
\]

Differentiate eqn with respect to \( y, z \). !

Other way for \( D_2 z(x, y) \): \( F(x, y, z(x, y)) = 0 \). To get \( D_2 z(x, y) \), differentiate to eqn with respect to \( y \):
\[
0 = D_1 F(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y) + D_2 F(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y)
\]
\[
+ D_3 F(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y)
\]
\[
\frac{\partial z}{\partial y}(x, y)
\]
\[
D_2 z(x, y)
\]