From last time; we are going to finish the computation of \(D^2z(x,y)\).

\[ \text{Ex} \quad \text{xyz} - z^2 + \sin(y) = 0 \] is supposed to define \(z\) implicitly as a function of \(x,y\).

To reduce chance of mistakes, write it as \(\text{xyz}(x,y) - z(x,y)^2 + \sin(y) = 0\).

\(z\) has two partial derivatives.

For \(D_x^1\) (to find \(z_x(x,y)\)): Differentiate eqn with respect to \(x\).
\[ yz_x(x,y) + xy D_x z(x,y) + 2z(x,y) D_z z(x,y) + 0 = 0 \] (New solve for \(z_x(x,y)\))

\(1\)st partial integral of \(y,z\).)

One way to \(D_x^2 z(x,y)\): \(F(x,y,z) = \) \(F(x,y,z(x,y)) = 0\). To get \(D_x^2 z(x,y)\), differentiate eqn with respect to \(y\): \(0 = 0, F(x,y,z(x,y)) + \frac{\partial F}{\partial x}(x,y,z(x,y)) \frac{\partial z}{\partial x} + \frac{\partial F}{\partial y}(x,y,z(x,y)) \frac{\partial z}{\partial y} + \frac{\partial F}{\partial z}(x,y,z(x,y)) \frac{\partial z}{\partial z} = 0 \).

Now fully start here.

\[ 0 = \frac{\partial F}{\partial x}(x,y,z(x,y)) + \frac{\partial F}{\partial y}(x,y,z(x,y)) \frac{\partial z}{\partial y} + \frac{\partial F}{\partial z}(x,y,z(x,y)) \frac{\partial z}{\partial z} = 0 \]

\(\text{New soln to soln}\) we want the two sides.

\[ D_x^2 z(x,y) = - \frac{\frac{\partial F}{\partial x}(x,y,z(x,y)) \frac{\partial z}{\partial y} + \frac{\partial F}{\partial z}(x,y,z(x,y)) \frac{\partial z}{\partial z}}{\frac{\partial F}{\partial x}(x,y,z(x,y))} \]

\(\text{Calculant}\) \(D_x^2 F(x,y,z(x,y)) = xz - 0 + \cos(y) = xz + \cos(y)\). So numerator is \(xz + \cos(y)\).

Do the denominator similarly.

\(\text{Here avoid } F_x(x,y,z(x,y))\) to avoid confusion with \(\frac{\partial}{\partial y} \left[ F(x,y,z(x,y)) \right] \)

What is wanted here is \(D_x^2 F(x,y,z(x,y))\) evaluated at \([x,y,z(x,y)]\) quite different.

\(\text{Directional derivatives.}\)

\(\text{Contour plot:}\)

\(\text{Let } f(x,y) \text{ be the function at } (x,y)\).

\(\text{At } (x_0,y_0) \text{ is the rate of change of elevation in the red direction.}\)

\(D_x^2 f(x_0,y_0) \text{ is the rate of change of elevation in the green direction.}\)

Does it make sense to consider the rate of elevation in the orange direction?

Yes. Its value will be positive but not as big as \(D_x^2 f(x_0,y_0)\).
Definition: Directional derivative in direction \( u = (a, b) \)

Look at the one variable function \( f(x) \) and find \( f'(x) \).

It is \( \lim_{h \to 0} \frac{f(x_0 + bu) - f(x_0)}{h} \).

This is \( f'(x_0) \).

It is called \( D_u f(x_0) \), if it exists.

Book restricts to choosing \( u \) to be a unit vector, which means you go in whatever direction with unit speed. Should not make this restriction.

\( D_u f(x_0, y_0) = D_u f(x_0, y_0) \) with \( u = \langle 1, 0 \rangle \). Similarly for \( D_2 f(x_0, y_0) \), take \( u = \langle 0, 1 \rangle \).

Example: \( f(x, y) = \begin{cases} \frac{x^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \)

Recall: Not cont. at \( (0, 0) \).

Since \( \lim_{t \to 0^+} f(t, 0) = 0 \) and \( \lim_{t \to 0^+} f(t, 0) = 0 \).

Find \( D_u f(0, 0) \) for all unit vectors \( u \). Take \( u = \langle \cos \theta, \sin \theta \rangle \).

For \( \theta \) in \((0, 2\pi)\), this runs through all unit vectors in \( \mathbb{R}^2 \).

We look at \( g_\theta(t) = f(t \cos \theta, t \sin \theta) \) and find \( g_\theta'(0) \).

We get: \( g_\theta(t) = \begin{cases} \frac{t \cos \theta - t^2 \cos^2 \theta \sin \theta}{t^2 \cos^2 \theta + t^2 \sin^2 \theta} & t \neq 0 \\ 0 & \ t = 0 \end{cases} \)

First, suppose \( \cos \theta \neq 0 \). Then \( g_\theta(t) = 0 \) for all \( t \). So \( g_\theta'(0) = 0 \).

Next, suppose \( \cos \theta = 0 \). Then

\[ g_\theta(t) = \frac{t \sin \theta}{t^2 \sin^2 \theta + t^2 \sin^2 \theta} \quad \text{for all } t. \]

(For \( t \neq 0 \), cancel \( t^2 \). But this is also right for \( t = 0 \).)

Does \( g_\theta'(0) \) exist? Yes: have a constant fraction of \( t \). Easy way to find it:

\[ g_\theta(t) = \lim_{h \to 0} \frac{g_\theta(0) - g_\theta(0)}{h} = \lim_{h \to 0} \frac{t \cos \theta \sin \theta}{(\cos^2 \theta + \sin^2 \theta)} \]
If $f$ is differentiable at $(x_0, y_0)$, then there is a formula: the directional derivatives are determined by the partial derivatives, by the linear approximation.

Written HW today: 10:15 pm. In 1 day, some answers will involve this info, but $h$ shall not appear since you know what it is. WebWork 8:10 pm. Next Friday, midterm 2. Next week, WebWork M=6, W=10, and F, no written HW.

Ex. $f(x, y) = x^2 + 2xy + 3y^2$, at $(1, 2)$, in direction $(\frac{3}{5}, \frac{4}{5}) = u$.

$D_u f(x, y) = D_1 f(x, y) v_1 + D_2 f(x, y) v_2$, where $u = (v_1, v_2)$.

Here, $D_u f(1, 2) = D_1 f(1, 2) (\frac{3}{5}) + D_2 f(1, 2) (\frac{4}{5})$.

$D_1 f(x, y) = 2x + 2y$, so $D_1 f(1, 2) = 6$, and $D_2 f(x, y) = 2x + 6y$, so $D_2 f(1, 2) = 14$.

$\nabla u \cdot D_u f(1, 2) = 6(\frac{3}{5}) + 14(\frac{4}{5}) = \frac{74}{5}$.

Lin approx says: If $t$ is close to 0, then

$f(1 + \frac{3}{5}t, 2 + \frac{4}{5}t) \approx f(1, 2) + D_1 f(1, 2)(\frac{3}{5})t + D_2 f(1, 2)(\frac{4}{5})t$

$= f(1, 2) + \left[ D_1 f(1, 2) (\frac{3}{5}) + D_2 f(1, 2) (\frac{4}{5}) \right] t$

$= 14 + (\frac{74}{5}) t$. This is the lin approx to $t \mapsto f(1 + \frac{3}{5}t, 2 + \frac{4}{5}t)$.

So the deriv at the $(x, y)$ must be

\[
\frac{74}{5} = D_1 f(1, 2) (\frac{3}{5}) + D_2 f(1, 2) (\frac{4}{5}),
\]