Remarks: see email from Saturday night (Friday night?): \( \leq 2 \) masses.


Canvas supposedly fully updated; except “extra practice HW2,” WebWork won’t be there.

Please make sure on my assignments.

Directly derivative in direction \( \mathbf{u} \). This works for any number of variables. If \( f \) is differentiable at \((x_1, \ldots, x_n)\), and \( \mathbf{u} \) is any vector in \( \mathbb{R}^n \), I need not be unit vector, but that case makes the most geometric sense: \( \mathbf{u} = \langle u_1, \ldots, u_n \rangle \). 

\[
D_\mathbf{u} f(x_1, \ldots, x_n) = D_1 f(x_1, \ldots, x_n) u_1 + D_2 f(x_1, \ldots, x_n) u_2 + \cdots + D_n f(x_1, \ldots, x_n) u_n.
\]

Yesterday’s ex wth \( \frac{x^3 + y^2}{x^2 + y^4} \) shows not true without differentiability.

Ex: \( f(x, y, z) = \cos(xy) + e^z \), deriv in dir \( (x, y, z) \) at point \((0, 1, \ln(2))\).

One could set \( g(t) = f(0 + \frac{1}{2} t, 3 + \frac{2}{3} t, \ln(2) - \frac{1}{4} t) \) and find \( g'(0) \).

Use partial der. \( \left( \frac{1}{2} \right) \text{(OK since \( f \) is differentiable)} \).

\[
D_1 f(x, y, z) = -\sin(xy) y, \quad D_2 f(x, y, z) = -\sin(xy) x, \quad D_3 f(x, y, z) = e^z.
\]

At \((0, 3, \ln(2))\), they are: \( \begin{array}{cc}
0 & 0 \\
0 & 2 \\
\end{array} \).

So \( D_\mathbf{u} f(0, 3, \ln(2)) = (0) \left( \frac{1}{2} \right) + (0) \left( \frac{2}{3} \right) - (2) (\frac{1}{4}) = -\frac{4}{8} \).

Ex: \( h(x, y, z) = x^2 + 3xy - yz - 4x^2 \) at \((3, 2, -1)\), direction \( \mathbf{v} = \langle \frac{3}{2}, -\frac{2}{3}, 1 \rangle \).

\[
h(x, y, z) = (3 + \frac{2}{3} t, 2 - \frac{2}{3} t, -1 + \frac{1}{3} t)
\]

\[
= (8 + \frac{2}{3} t)^2 + 3(3 + \frac{2}{3} t)(2 - \frac{2}{3} t) - (-1 + \frac{1}{3} t) - 4(3 + \frac{2}{3} t)(-1 + \frac{1}{3} t)
\]

Now need \( g'(0) \). Omit.

Partial der., \( D_1 h(x, y, z) = 2x + 3y - 4z \) \( \begin{array}{cc}
16 & 16 \\
9 & 9 \\
-13 & -13
\end{array} \) at \((3, 2, -1)\).

So \( D_\mathbf{v} h(3, 2, -1) = 16 \left( \frac{3}{2} \right) + 9 \left( -\frac{2}{3} \right) + (-13) \left( \frac{1}{3} \right) = \frac{32 - 18 - 13}{3} = \frac{1}{3} \).

Why did restrict to unit vectors? If \( f \) is diff at \((x_0, y_0)\), then \( D_{\mathbf{v}_1 \mathbf{v}_2} f(x_0, y_0) = D_{\mathbf{v}_1} f(x_0, y_0) + D_{\mathbf{v}_2} f(x_0, y_0) \). (Even if \( \mathbf{v}_1, \mathbf{v}_2 \) are unit vectors; \( \mathbf{v}_1 + \mathbf{v}_2 \) usually isn't.)
Define the gradient $\text{grad}(f) \equiv \nabla f$ ("del f"), by
\[
\text{grad}(f)(x, y, z) = (D_1 f(x, y, z), D_2 f(x, y, z), D_3 f(x, y, z)),
\]
(Similarity for any number of variables).

Get formula: $D_u f(x, y, z) = \text{grad}(f)(x, y, z) \cdot u.
\]

**Example:**
\[
f(w, x, y, z) = w + \cos(x^2 + xz) + e^{yz}.
\]
\[
\text{grad}(f)(x, y, z) = \left(1, -\sin(x^2 + xz)(2x + z), ze^{yz}, -\sin(x^2 + xz)x + ye^{yz}\right),
\]

\[
f_w(w, x, y, z) \neq f_x(w, x, y, z)
\]

Take dir. deriv in direction $v = (-1, -2, -3, -4)$ at $(w, x, y, z)$

\[
D_v f(w, x, y, z) = \text{grad}(f)(w, x, y, z) \cdot v
\]

\[
= (1)(-1) + [-\sin(x^2 + xz)(2x + z)](-2) + (ze^{yz})(-3) + [-\sin(x^2 + xz)x + ye^{yz}](4)
\]

\[
= 0 0 + 0 + 0 + 0
\]

Could also take $(w, x, y, z) = (2, -1, 3, -2)$, for example.

**Geometric significance:** $\text{grad}(f)(x, y)$ is in the direction in which $f$ increases the fastest.

- $\text{grad}(f)(x, y)$ is usually not a unit vector.
- $D_u f(x, y) = \text{grad}(f)(x, y) \cdot u = \|\text{grad}(f)(x, y)\| ||u|| \cos \theta$, for fixed $u$.

Get largest value if $\cos(\theta) = 1$, that is, $\theta = 0$, that is, $\text{grad}(f)(x, y)$ and $u$ are parallel and point in same way.

If make this choice, $D_u f(x, y) = \|\text{grad}(f)(x, y)\|$, then $f$ is increasing in this direction.

**Example:**
You are on a hill. Elevation at $(x, y)$ is $h(x, y) = 2000 - x^2 - 2xy - 2y^2$ (in meters, $x, y$ measured in 100's of meters).

Questions: If not to go up/down as fast as possible, what direction do you go?

If want to stay at same elevation, what direction?

What are tangent line and normal line to contour? $A \hat{f} (1, 2)$.