Example from last time (with contour plot at right):

\[ h(x,y) = 2x + x^2 - 2xy - y^2, \]  
height of a hill at position \((x,y)\). (For units, take \(h(x,y)\) in meters, \(x,y\) in 100's of meters).

You are at \((-1,2)\).

Direction of steepest ascent? Descend? How fast?

Directions in which \(z\) not change elevation?

Equations of tangent and normal lines to the contour at this point?

From last time: \(\text{Need grad } h\) \((-1,2)\).

\[\nabla h(x,y) = \begin{bmatrix} -1 - 2x - 2y \end{bmatrix}, \quad \frac{\partial h}{\partial x}(-1,2) = -2, \quad \frac{\partial h}{\partial y}(-1,2) = -4.\]

\[\text{grad } h(-1,2) = \langle -3, -6 \rangle = 3 \langle -1, -2 \rangle.\]

What a \(\text{unit vector in this direction}\). \(|| \text{grad } h(-1,2) || = 3 \sqrt{5} \). \(\text{So a unit vector in this direction is} \) \(\frac{1}{3 \sqrt{5}} \langle -3, -6 \rangle = \langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle\).

Rate of increase is \(|| \text{grad } h(-1,2) || = 3 \sqrt{5} \) \(\text{in meters per hundred meters}\).

Steepest decrease? \(\text{Exactly the opposite direction, so unit vector} \) \(\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle\).

\(\text{From whenever the function is differentiable}\).

Tangent line to level curve is in the directions we go to stay at same elevation, that is, \(\text{it is unit to follow the level curve}. \) \(\text{If} \ u = \langle u_1, u_2 \rangle \text{ is a unit vector in this direction, then we unit}\)

\[\nabla h(-1,2) = 0. \text{This is} \ D_1 h(-1,2) u_1 + D_2 h(-1,2) u_2 = \text{grad } h(-1,2) \cdot u\]

\(\text{It says} \ u \text{ shall be orthogonal to the gradient}. \) \text{For direction: non-unit vector, take} \ \langle 6, -3 \rangle \text{, getting unit vector} \ \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle. \text{The other direction} - \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle.

Equation of tangent line: \(\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle (x+1) - 2(y-2) = 0\)

\(\text{could use} \ \langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle \text{ or} \ (-1, -2)\)

Expand it, \(-2(x+1) - 6(y-2) = 0\). \text{Omit simplification}.\n
\(\text{Parametric equations: use direction in the line,} \ x(t) = -1 + 6t, y(t) = 2 - 3t. \)

\(\text{Parametric equations for normal line:} \ x(t) = -1 - 3t, y(t) = 2 - 6t.\)
Ex. You are a mosquito looking for something to bite by following the relative humidity
Fly in direction of largest increase of RH. Suppose RH is given by
\[ R(x, y, z) = 10 + \frac{10}{2 + x^2 + xy + 4y^2 + 2z^2} \]
You are at \((3, 1, -1)\).

Want to find \(\nabla R(3, 1, -1)\).

Need \(D_1 R(3, 1, -1) = -\frac{10(2x)}{(2 + x^2 + xy + 4y^2 + 2z^2)}\)
\[ D_2 R(3, 1, -1) = -\frac{10(x - 8y)}{(2 + x^2 + xy + 4y^2 + 2z^2)} \]
\[ D_3 R(3, 1, -1) = \frac{-10(4z)}{(2 + x^2 + xy + 4y^2 + 2z^2)} \]

We need \(\nabla R(3, 1, -1) = \langle D_1 R(3, 1, -1), D_2 R(3, 1, -1), D_3 R(3, 1, -1) \rangle\).

I got \(\langle \frac{-7}{40}, \frac{-11}{40}, \frac{4}{40} \rangle = \frac{1}{40} \langle -7, -11, 4 \rangle\). Unit vector \(\vec{u}\):
\[ \left(\frac{1}{\sqrt{186}}\right) \langle -7, -11, 4 \rangle = \frac{1}{\sqrt{186}} \langle -7, -11, 4 \rangle \]
This is the right direction.

Next: What is the equation of the tangent plane to level surface at \((3, 1, -1)\)?

Note: Since in 3 dim, have a level surface rather than a level curve.

It has a tangent plane rather than a tangent line.

Directions in tangent plane should be (they are) orthogonal to the gradient.

Equation will be
\[ \nabla R(3, 1, -1) \cdot \langle x - 3, y - 1, z + 1 \rangle = 0 \]
\[ \frac{1}{40} \left( -7(x - 3) - 11(y - 1) + 4(z + 1) \right) = 0 \]

Simplify: \(-7(x - 3) - 11(y - 1) + 4(z + 1) = 0\). (Can simplify further).

Rewrite: Directions in tangent plane are those directions \(\vec{w}\) with \(D_3 R(3, 1, -1) \cdot \vec{w} = 0\).

That is \(\nabla R(3, 1, -1) \cdot \vec{w} = 0\). Gives eventually the same equation.

The level surface has a normal line

Parametrization: \(x(t) = 3 - \frac{7}{40} t, y(t) = 1 - \frac{11}{40} t, z(t) = -1 + \frac{4}{40} t\).

Same line with \(x(t) = 3 - 7t, y(t) = 1 - 11t, z(t) = -1 + 4t\).