Suppose a function \( f \) of \( n \) variables has a critical point at \( \mathbf{a} = (a_1, a_2, \ldots, a_n) \). There will be a "special direction" associated with this critical point, say the first is the direction in which \( f \) curves up the sharpest; the next sharpest after that direction is removed; I among the directions orthogonal to the first, etc., ending with the direction in which \( f \) curves down the sharpest.

For those who have seen linear algebra: These directions are eigenvectors of the Hessian matrix whose \( i \)-th entry is \( \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) \) (so it's adjoint by Clairaut's Theorem), and the corresponding eigenvalues, which is positive if \( f \) curves up, negative if \( f \) curves down. Here is a local min. if all eigenvalues are \( >0 \), max if all are \( <0 \), and a kind of generalized saddle point if some are strictly positive and some are strictly negative.

The determinant of a matrix is the product of its eigenvalues. In two dimensions, if \( \det(\mathbf{f}) < 0 \), the eigenvalues have opposite signs, so get a saddle point. If \( \det(\mathbf{f}) > 0 \), they have the same sign, so have a local min or max. (No such shortcut even in dimension 3.)

**Additional comments:** In one variable, \( f'(a) = 0 \) and \( f''(a) \neq 0 \), then \( f \) curves up or down depending on sign of \( f''(a) \). In several variables, \( f \) can do different things in different directions. If \( f \) curves up in all directions, have \( \text{local min} \). If \( f \) curves down in some directions and up in others, have a generalized saddle.

In two variables: One can find direction in which \( f \) curves up, and direction with sharpest curve down (least slope up) \( \left( \text{all both up, or both down, \(-\)} \right) \). Special is two dimensions:

\[
\det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \det \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix} = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2
\]

- If \( \det(\mathbf{f}) > 0 \), the both directions curve up or both down \( [\text{eigenvalues have same sign}] \), so local min or max. Further cheap trick: if \( f_{xx}(a,b) > 0 \), local min; if \( f_{xx}(a,b) < 0 \), local max.

[Could use \( f_{xy}(a,b) \) instead.]

Continue: If \( \det(\mathbf{f}) < 0 \), get up in one direction, down in another -- saddle pt.
Examples for Monday

Ex 1: \( f(x, y) = x^2 + xy + y^2 + y \), \( D_f(x, y) = 2x + y \), \( D^2 f(x, y) = 1 \), \( D^2 f(x, y) = 2 \). So \( D\left(\frac{1}{3}, -\frac{2}{3}\right) = \text{det}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 2(2) - 1^2 = 3 > 0 \)

Since \( D^2 f(x, y) > 0 \), have a local minmum.

Ex 2: \( g(x, y) = y \sin(x) \), \( D_g(x, y) = y \cos(x) \), \( D^2 g(x, y) = \sin(x) \), crit. pts \( (n\pi, 0) \) for any integer \( n \).

\( D^2 g(x, y) = -y \sin(x) \), \( D^2 g(x, y) = \cos(x) \), \( D^2 g(x, y) = 0 \).

At \( (n\pi, 0) \): \( D^2 g(n\pi, 0) = 0 \), \( D^2 g(x, y) = \cos(n\pi) \), which is \( \pm 1 \) depending on whether \( n \) is even or odd. Look at \( \text{det}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) or \( \text{det}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Both are \(-1\), so all crit. pts are saddle pts.

Ex 3: \( h(x, y) = x^3 - 3x + 3xy^2 \). It had no critical points. (Saddle pts)

Ex 4: \( k(x, y) = x^3 + y^3 \). \( D_k(x, y) = 3x^2 + 3y^2 \), \( D^2 k(x, y) = 3(y^2) \), critical point \( (0, 0) \).

\( D^2 k(x, y) = 6x \), \( D^2 k(x, y) = 0 \), \( D^2 k(x, y) = 6y \). At \( (0, 0) \), all of these are zero. So \( D(0, 0) = 0 \).

Test gives no conclusion. Here none of local min, local max, or saddle pt. But could have any of these.

Ex 5: \( h(x, y) = x^4 + y^4 - 4xy + 1 \), \( D_1 h(x, y) = 4x^3 - 4y \), \( D_2 h(x, y) = 4y^3 - 4x \). Cost pts \( (0, 0) \) \( (1, 1) \) \( (1, -1) \).

Omit, (Cost saddle pt at \( (0, 0) \) and local mins at the other two points.)

Find and classify critical pts.
Blue is low, (about \(-4\)) and
Red is high (about \(4\)).

Critical pts at:
\(-1, 1\) : local min
\(-1, -1\) : local min
\((1, 0)\) : local max
\((1, 1)\) : saddle pt
\((1, -1)\) : saddle pt
\((-1, 0)\) : saddle pt
Homework: Due today 3-4 pm: one person. Next two.

Other times: Tuesday 8:30-9:30 pm

Tomorrow 11 am - noon, 2-3 pm.

Friday

Global min max

Theorem (not here). If $f$ is continuous on a bounded (contains in some ball, does not go off to 0) closed (contains its boundary; like $[a,b]$ in $\mathbb{R}$) subset $D$ of $\mathbb{R}^n$, then $f$ has a global min and a global max. on $D$. If they occur in interior (not on boundary), they must occur at critical points.

Ex: Find points on the cone $z^2 = x^2 + y^2$ closest to $(4,2,0)$.

Want to minimize $\text{dist} \ (x,y,z), (4,2,0) = ||(x,y,z)-(4,2,0)|| \quad \text{subject to } \ z^2 = x^2 + y^2$.

Cheap trick: Minimize $||(-x,y,0)||^2 = x^2 + y^2$ that is, minimize $(x-4)^2 + (y-2)^2 + (z-0)^2$ for $(x,y,z) \in \mathbb{R}$. Domain is not bounded.

The min distance is at most $\text{dist} (0,0,0), (4,2,0) < 6$, so enough to consider $|x| \leq 6, |y| \leq 6$. This is a closed ball domain.

The plan: Find all critical pts of $d(x,y) = (x-4)^2 + (y-2)^2 + x^2 + y^2$, evaluate $d(x,y)$ at each of them, and choose the min with smallest value of $d(x,y)$.

Then solve for $z$, for each of these.