MATH 281 (FALL 2020, PHILLIPS): COMMENTS ON AND SOLUTIONS TO HOMEWORK 2

1. Comments on mistakes

There are **still** files being turned in with spaces or other improper characters in their names. There are also **still** people writing ~~$3 \cdot -27$~~.

Other mistakes:

Vectors on the right in scalar multiplication: If c is a scalar and $\mathbf{v}$ is a vector, the scalar product of c and $\mathbf{v}$ is always $c\mathbf{v}$, *never* ~~$\mathbf{v}c$~~. This applies even when using formal determinants to calculate cross products. Thus, the formal determinant of

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -2 \\ 2 & -4 & -5 \end{pmatrix}$$

is

$$[(3)(-5) - (-2)(-4)]\mathbf{i} - [(-1)(-5) - (-2)(2)]\mathbf{j} + [(-1)(-4) - (3)(2)]\mathbf{k},$$

not

$$\cancel{\mathbf{i}[(3)(-5) - (-2)(-4)]} - \cancel{\mathbf{j}[(-1)(-5) - (-2)(2)]} + \cancel{\mathbf{k}[(-1)(-4) - (3)(2)]}.$$

Scalar multiplication can't be written as divsion: If $\mathbf{v}$ is a vector, and c is a scalar, the expressions

$$\cancel{\frac{\mathbf{v}}{c}} \qquad \text{and} \qquad \cancel{\mathbf{v}/c}$$

are both wrong. The correct way to write these is as scalar multiplication:

$$\left(\frac{1}{c}\right)\mathbf{v} \qquad \text{or} \qquad c^{-1}\mathbf{v}.$$

The vector *always* goes to the right!

The parentheses are not strictly necessary in the first expression, but they help keep down confusion.

Use of equals signs: As stated in the solutions to Written Homework 1, write "=" when you are claiming two expressions are equal. Do **not** write "=" between two expressions which are not are equal. Example:

$$\cancel{-23\mathbf{i} - 9\mathbf{j} - 2\mathbf{k} = \sqrt{(-23)^2 + (-9)^2 + (-2)^2}}.$$

Among other things, one of these is a vector and the other is a scalar.

Say what things are: In Problem 3 Part (1), you may not write just:

Date: 9 October 2020.

We need to solve the equation $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$. This means that

$$\langle (2)(z) - (1)(y),\ (1)(x) - (1)(z),\ (1)(y) - (2)(x)\rangle = \langle 3, 1, -5\rangle.$$

Therefore $2z - y = 3, \ldots$.

You **must** say what x, y, and z are. Correctly written:

We need to solve the equation $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$. **Write** $\mathbf{v} = \langle x, y, z\rangle$**.** Then

$$\langle (2)(z) - (1)(y),\ (1)(x) - (1)(z),\ (1)(y) - (2)(x)\rangle = \langle 3, 1, -5\rangle.$$

Therefore $2z - y = 3, \ldots$.

Matrices and determinants: The expressions

$$\begin{pmatrix} 2 & t & -3 \\ -1 & 3 & -y \\ -2 & -4 & 4 \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & t & -3 \\ -1 & 3 & -y \\ -2 & -4 & 4 \end{bmatrix} \tag{1}$$

are both commonly used for a matrix. (I usually use the first, but either is fine. I can't find either of them in the book.) However, they are **never** used for the determinant of anything. The notation

$$\begin{vmatrix} 2 & t & -3 \\ -1 & 3 & -y \\ -2 & -4 & 4 \end{vmatrix}$$

is (unfortunately) often used for the determinant of the matrix in (1). However, it is **never** used for the matrix itself. Please keep these straight!

Check that supposed solutions really are solutions: In Problem 3 Part (1), it isn't clear that you have used all the information you can get out of the equations. So you need to check that your claimed possible vectors $\mathbf{v}$ actually satisfy

$$\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle.$$

Don't use "=" for approximations: In Problem 2 Part (2), it is wrong to write ~~$\sqrt{614} = 24.779$~~. The correct version is $\sqrt{614} \approx 24.779$.

$\|\cdot\|$ is not used for absolute value: The notation $\|\mathbf{v}\|$ for the norm (or magnitude) of a vector is much more common in mathematics than $|\mathbf{v}|$, for the purpose of distinguishing it from the absolute value of a scalar. For a scalar c, the notation ~~$\|c\|$~~ is wrong.

2. Solutions

This homework assignment is due Friday 9 Oct. 2020 at 10:00 pm, to be uploaded as a pdf file (or one of a few other allowed file types) on the University of Oregon Canvas site.

General instructions: show work in all problems, and be very careful to use fully correct notation. Incorrect notation will lose credit on exams (grading is based on what you write, not what you meant), and the written homework assignments are your chance to have me tell you whether your notation is correct.

Files turned in must have good enough resolution that I can read them easily.

Apart from the extension (such as "`.pdf`"), your file name should contain only numbers, capital and lowercase letters, and underscores. In particular, **no** spaces or parentheses.

Write your name on all pages.

Problem 1 (3 points)**.** Find

$$\det \begin{pmatrix} 2 & t & -3 \\ -1 & 3 & -y \\ -2 & -4 & 4 \end{pmatrix}.$$

Solution.

$$\det \begin{pmatrix} 2 & t & -3 \\ -1 & 3 & -y \\ -2 & -4 & 4 \end{pmatrix}$$
$$= 2 \det \begin{pmatrix} 3 & -y \\ -4 & 4 \end{pmatrix} - t \det \begin{pmatrix} -1 & -y \\ -2 & 4 \end{pmatrix} + (-3) \det \begin{pmatrix} -1 & 3 \\ -2 & -4 \end{pmatrix}$$
$$= (2)[(3)(4) - (-y)(-4)] - (t)[(-1)(4) - (-y)(-2)] + (-3)[(-1)(-4) - (3)(-2)]$$
$$= 2[12 - 4y] - t[-4 - 2y] - 3[4 + 6]$$
$$= 2ty + 4t - 8y - 6.$$

□

Problem 2 (12 points)**.** Define vectors in $\mathbb{R}^3$ by

$$\mathbf{u} = \langle 1, 4, -3 \rangle, \qquad \mathbf{v} = \langle -1, 3, -2 \rangle, \qquad \text{and} \qquad \mathbf{w} = \langle 2, -4, -5 \rangle.$$

Find:

(1) $\mathbf{v} \times \mathbf{w}$.

Solution. This is the formal determinant of

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -2 \\ 2 & -4 & -5 \end{pmatrix},$$

which is

$$[(3)(-5) - (-2)(-4)]\mathbf{i} - [(-1)(-5) - (-2)(2)]\mathbf{j} + [(-1)(-4) - (3)(2)]\mathbf{k}$$
$$= (-15 - 8)\mathbf{i} - (5 + 4)\mathbf{j} + (4 - 6)\mathbf{k}.$$
$$= -23\mathbf{i} - 9\mathbf{j} - 2\mathbf{k}.$$

You can also write

$$\langle -1, 3, -2 \rangle \times \langle 2, -4, -5 \rangle = \langle -23, -9, -2 \rangle.$$

□

(2) The area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

Solution. This is (using the result of Part (1) at the first step)

$$\|\mathbf{v} \times \mathbf{w}\| = \sqrt{(-23)^2 + (-9)^2 + (-2)^2} = \sqrt{614}.$$

There is no need to give a numerical approximation to $\sqrt{614}$. □

(3) The volume of the parallelopiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Solution. This is (using the result of Part (1) at the first step)

$$\begin{aligned} |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| &= \big|\langle 1,\,4,\,-3\rangle\cdot\langle -23,\,-9,\,-2\rangle\big| \\ &= (1)(-23)+(4)(-9)+(-3)(-2) = |-53| = 53. \end{aligned}$$

□

(4) All unit vectors orthogonal to both **u** and **w**.

Solution. We first find some vector **x** orthogonal to both **u** and **w**. The obvuious choice is $\mathbf{u}\times\mathbf{w}$, which is the formal determinant of

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -3 \\ 2 & -4 & -5 \end{pmatrix}.$$

Thus

$$\begin{aligned} \mathbf{u}\times\mathbf{w} &= [(4)(-5)-(-3)(-4)]\mathbf{i} - [(1)(-5)-(-3)(2)]\mathbf{j} + [(1)(-4)-(4)(2)]\mathbf{k} \\ &= (-20-12)\mathbf{i} - (-5+6)\mathbf{j} + (-4-8)\mathbf{k}. \\ &= -32\mathbf{i} - \mathbf{j} - 12\mathbf{k}. \end{aligned}$$

Now we need find all unit vectors parallel to this direction. There are two choices:

$$\|\mathbf{u}\times\mathbf{w}\|^{-1}\mathbf{u}\times\mathbf{w} \qquad \text{and} \qquad -\|\mathbf{u}\times\mathbf{w}\|^{-1}\mathbf{u}\times\mathbf{w}.$$

So we need

$$\|\mathbf{u}\times\mathbf{w}\| = \sqrt{(-32)^2+(1)^2+(-12)^2} = \sqrt{1169}.$$

Now the two choices are

$$\left\langle -\frac{32}{\sqrt{1169}},\, -\frac{1}{\sqrt{1169}},\, -\frac{12}{\sqrt{1169}} \right\rangle \qquad \text{and} \qquad \left\langle \frac{32}{\sqrt{1169}},\, \frac{1}{\sqrt{1169}},\, \frac{12}{\sqrt{1169}} \right\rangle.$$

□

Problem 3 (10 points)**.**

(1) Find all vectors **v** such that

$$\langle 1,\,2,\,1\rangle\times\mathbf{v} = \langle 3,\,1,\,-5\rangle.$$

Solution. Write $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then calculate:

$$\begin{aligned} \langle 1,\,2,\,1\rangle\times\mathbf{v} &= \langle (2)v_3-(1)v_2,\, (1)v_1-(1)v_3,\, (1)v_2-(2)v_1\rangle \\ &= \langle 2v_3-v_2,\, v_1-v_3,\, v_2-2v_1\rangle. \end{aligned}$$

Therefore, if

$$\langle 1,\,2,\,1\rangle\times\mathbf{v} = \langle 3,\,1,\,-5\rangle,$$

then

$$2v_3-v_2=3, \qquad v_1-v_3=1, \qquad \text{and} \qquad v_2-2v_1=-5.$$

We can solve the second and third of these for v_1, getting

$$v_2 = 2v_1-5 \qquad \text{and} \qquad v_3 = v_1-1.$$

Therefore any vector **v** as requested must be of the form

$$\mathbf{v} = \langle v_1,\, 2v_1-5,\, v_1-1\rangle$$

for some real number v_1.

We need to check that these vectors actually all are solutions to the problem. So we calculate:

$$\begin{aligned}
&\langle 1, 2, 1\rangle \times \langle v_1, 2v_1 - 5, v_1 - 1\rangle \\
&\quad = \langle (2)(v_1 - 1) - (1)(2v_1 - 5), (1)(v_1) - (1)(v_1 - 1), (1)(2v_1 - 5) - (2)(v_1)\rangle \\
&\quad = \langle 3, 1, -5\rangle,
\end{aligned}$$

which is what it is supposed to be.

One could have solved for v_2 or v_3 instead. □

The check at the end is necessary. Since it isn't clear you used all the information to be gotten from the equations, you otherwise don't know that these vectors actually all are solutions.

(2) Explain, without trying to solve equations, why there is no vector **w** such that

$$\langle 1, 2, 1\rangle \times \mathbf{w} = \langle 3, 1, 5\rangle.$$

Solution. If $\mathbf{v} \times \mathbf{w} = \mathbf{r}$, then we know that $\mathbf{r}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Therefore $\mathbf{v} \cdot \mathbf{r} = 0$ and $\mathbf{w} \cdot \mathbf{r} = 0$. Here, taking

$$\mathbf{v} = \langle 1, 2, 1\rangle \qquad \text{and} \qquad \mathbf{r} = \langle 3, 1, 5\rangle,$$

we get

$$\mathbf{v} \cdot \mathbf{r} = \langle 1, 2, 1\rangle \cdot \langle 3, 1, 5\rangle = 10 \neq 0,$$

so no such **w** can exist. □