MATH 281 (FALL 2020, PHILLIPS): COMMENTS ON AND SOLUTIONS TO HOMEWORK 3

1. Comments on mistakes

1.1. **Miscellaneous.** On Problem 4, many people drew only the part of the curve one gets for $t \geq 0$, which is only the bottom half of it. Some people graphed $\mathbf{r}'(t)$ instead of $\mathbf{r}(t)$. Many people didn't label their axes, didn't put scales on them, or both.

The symbols "$\to$" and "$\Longrightarrow$" are not synonyms for "$=$". The computations

$$\lim_{t\to 2} \frac{t^2-4}{t-2} \Longrightarrow \lim_{t\to 2} \frac{(t-2)(t+2)}{t-2} \Longrightarrow \lim_{t\to 2}(t+2) \Longrightarrow 4$$

and

$$\lim_{t\to 2} \frac{t^2-4}{t-2} \to \lim_{t\to 2} \frac{(t-2)(t+2)}{t-2} \to \lim_{t\to 2}(t+2) \to 4$$

are not correctly written. The right form is:

$$\lim_{t\to 2} \frac{t^2-4}{t-2} = \lim_{t\to 2} \frac{(t-2)(t+2)}{t-2} = \lim_{t\to 2}(t+2) = 4.$$

If you want to take the derivative of $-t^3$, parentheses are required:

$$\frac{d}{dt}\left(-t^3\right).$$

The expressioon

$$\frac{d}{dt} - t^3$$

does have a meaning, but the meaning is something else.

Date: 22 October 2020.

1.2. **Limit notation.** The expression "$\lim_{x\to a}$" (or other limit) must be present when the limit remains to be taken, and must not be present after the limit has been taken. Thus, both

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\lim_{x\to 2}\frac{(x+7)(x-2)}{x-2}=\lim_{x\to 2}(x+7)=\cancel{\lim_{x\to 2}}(2+7)=9$$

and

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\cancel{\frac{(x+7)(x-2)}{x-2}}=\cancel{x+7}=2+7=9$$

are wrong—with errors at the crossed out expressions. The only correct placement of "$\lim_{x\to 2}$" in this calculation is as shown here:

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\lim_{x\to 2}\frac{(x+7)(x-2)}{x-2}=\lim_{x\to 2}(x+7)=2+7=9.$$

Putting "$\lim_{x\to a}$" where it belongs, and not where it doesn't belong, is part of understanding what you are doing in this course.

No expression of the form

$$\cancel{\lim_{x\to 2}=9}$$

is ever correct. One may never have "=" (or any operation or relation symbol) directly after a limit symbol. One must always take the limit *of something.*

Parentheses are needed when taking the limit of a sum. (You have already seen some rules about the order of operations, such as multiplication before addition. Here are more: multiplication and division before limits, and limits before addition and subtraction.)

For example, it is not correct to write "$\cancel{\lim_{x\to 2} x+7}$" instead of "$\lim_{x\to 2}(x+7)$" in the computation

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\lim_{x\to 2}\frac{(x+7)(x-2)}{x-2}=\lim_{x\to 2}(x+7).$$

Similarly, the notation

$$\cancel{\lim_{h\to 0} 1-2x-h}$$

is not correct. It means

$$\left(\lim_{h\to 0} 1\right)-2x-h,$$

which does not make sense. Write

$$\lim_{h\to 0}\big(1-2x-h\big).$$

You need parentheses even in

$$\lim_{h\to 0}\big(-h\big).$$

2. Solutions

This homework assignment is due Friday 16 Oct. 2020 at 10:00 pm, to be uploaded as a pdf file (or one of a few other allowed file types) on the University of Oregon Canvas site.

General instructions: show work in all problems, and be very careful to use fully correct notation. Incorrect notation will lose credit on exams (grading is based on what you write, not what you meant), and the written homework assignments are your chance to have me tell you whether your notation is correct.

Files turned in must have good enough resolution that I can read them easily.

Apart from the extension (such as "`.pdf`"), your file name should contain only numbers, capital and lowercase letters, and underscores. In particular, **no** spaces or parentheses. You do not need to put any identifying information in the file name; "`HW3.pdf`" is quite sufficient. (The Canvas system adds enough identifying information.)

Write your name on all pages.

Problem 1 (5 points)**.** Determine where the lines given by the parametric equations

$$\mathbf{r}_1(t) = \langle 1, 1, 0 \rangle + t\langle 1, \, -1, \, 2 \rangle \qquad \text{and} \qquad \mathbf{r}_2(t) = \langle 2, 0, 2 \rangle + t\langle -1, \, 1, \, 0 \rangle$$

intersect. Be sure your claimed intersection point really is one!

Solution. We solve the equation

$$\langle 1, 1, 0 \rangle + t\langle 1, \, -1, \, 2 \rangle = \langle 2, 0, 2 \rangle + s\langle -1, \, 1, \, 0 \rangle$$

for $s, t \in \mathbb{R}$. (If the lines intersect, there is no reason to suppose that the same value of the parameter occurs for both at the intersection point.) In coordinates, we get three equations:

$$1 + t = 2 - s, \qquad 1 - t = -s, \qquad \text{and} \qquad 2t = 2.$$

The last implies $t = 1$, and then the second implies $s = 0$.

Since it isn't clear that we used all the information, we need to check that we really have a solution. We have

$$\mathbf{r}_1(1) = \langle 1, 1, 0 \rangle + \langle 1, \, -1, \, 2 \rangle = \langle 2, 0, 2 \rangle \qquad \text{and} \qquad \mathbf{r}_2(0) = \langle 2, 0, 2 \rangle.$$

These agree, so the lines really do intersect, at $\langle 2, 0, 2 \rangle$. □

Remark. Here is why we need to check. Suppose the problem said, "Determine where the lines given by the parametric equations

$$\mathbf{r}_1(t) = \langle 114, 1, 0 \rangle + t\langle 1, \, -1, \, 2 \rangle \qquad \text{and} \qquad \mathbf{r}_2(t) = \langle 2, 0, 2 \rangle + t\langle -1, \, 1, \, 0 \rangle$$

intersect." (The only difference is that the first coordinate of the first vector has been changed from 1 to 114.) Our equations are now

$$114 + t = 2 - s, \qquad 1 - t = -s, \qquad \text{and} \qquad 2t = 2.$$

As before, the last implies $t = 1$, and then the second implies $s = 0$. But now

$$\mathbf{r}_1(1) = \langle 115, 0, 2\rangle \qquad \text{and} \qquad \mathbf{r}_2(0) = \langle 2, 0, 2\rangle.$$

These don't agree, so we didn't actually find a solution. □

Problem 2 (8 points)**.** Describe and sketch the surface

$$x^2 + \frac{y^2}{4} = z^2 - 2z.$$

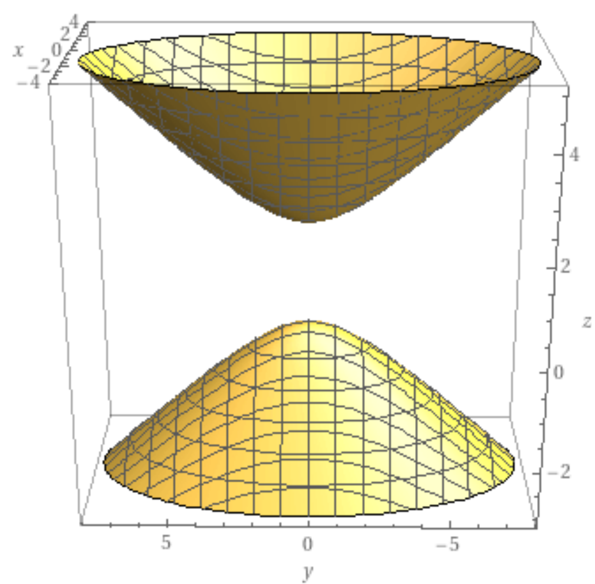
Be sure to describe the traces in planes parallel to each the the coordinate planes.

Solution. We complete the square rearrange and:

$$x^2 + \frac{y^2}{4} = (z^2 - 2z + 1) - 1 = (z-1)^2 - 1$$

$$-x^2 - \frac{y^2}{4} + (z-1)^2 = 1.$$

This is a two sheeted hyperboloid which opens up and down (along the z-axis), and has center at $(0, 0, 1)$ and vertices at $(0, 0, 2)$ and $(0, 0, 0)$. Here is the graph:



The horizontal traces have equations of the form $x^2 + \frac{y^2}{4} = k$ for $k = (z-1)^2 - 1$. If $0 < z < 2$, then k is negative, and the trace is empty. If $z = 0$ or $z = 2$, then $k = 0$, and the trace is the single point $(0, 0)$. If $z > 2$ or $z < 0$, then $k > 0$, so the trace is an ellipse, centered at $(0, 0)$, and with the longer axis in the y direction.

The traces in planes parallel to the xz plane have equations of the form $(z-1)^2 - x^2 = k$ with $k = 1 + \frac{y^2}{4}$. Since $k > 0$ for all real y, these are all hyperbolas opening up and down. Similarly, the traces in planes parallel to the xy plane have equations of the form $(z-1)^2 - \frac{y^2}{4} = k$ with $k = 1 + x^2$. Since $k > 0$ for all real z, these are also all hyperbolas opening up and down. □

Problem 3 (6 points)**.** Find

$$\lim_{t \to 2} \left\langle \frac{t^2 - 4}{t - 2}, \ \sqrt{t+7}, \ \frac{\sin(t-2)}{6t - 12} \right\rangle.$$

Simplify your answer.

Solution. The limit is evaluated coordinatewise. The limits of the first and third coordinates have the indeterminate form "$\frac{0}{0}$". We calculate:

$$\lim_{t \to 2} \frac{t^2 - 4}{t - 2} = \lim_{t \to 2} \frac{(t-2)(t+2)}{t - 2} = \lim_{t \to 2} (t + 2) = 4.$$

For the third coordinate, so we try L'Hopital's Rule:

$$\lim_{t \to 2} \frac{\sin(t-2)}{6t - 12} = \lim_{t \to 2} \frac{\cos(t-2)}{6} = \frac{1}{6}.$$

Clearly $\lim_{t \to 2} \sqrt{t+7} = \sqrt{9} = 3$. Therefore

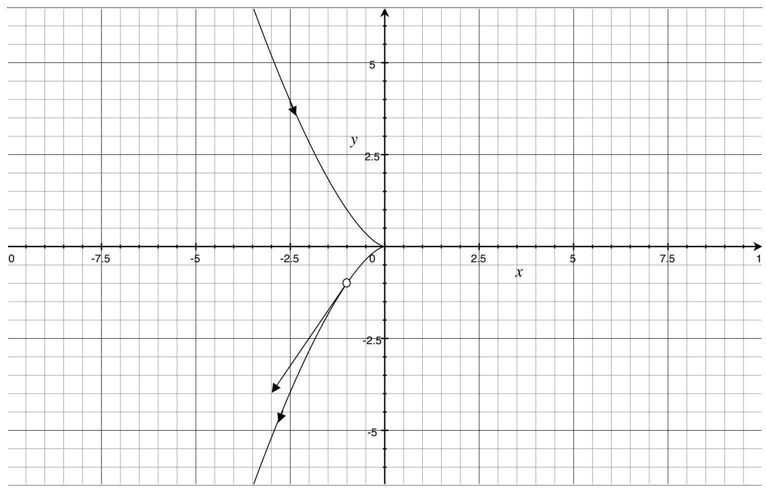
$$\lim_{t \to 2} \left\langle \frac{t^2 - 4}{t - 2}, \ \sqrt{t+7}, \ \frac{\sin(t-2)}{6t - 12} \right\rangle = \left\langle \lim_{t \to 2} \frac{t^2 - 4}{t - 2}, \ \lim_{t \to 2} \sqrt{t+7}, \ \lim_{t \to 2} \frac{\sin(t-2)}{6t - 12} \right\rangle = \left\langle 4, \ 3, \ \frac{1}{6} \right\rangle.$$

□

Problem 4 (6 points)**.** Consider the parametrized plane curve $\mathbf{r}(t) = \langle -t^2, \ -t^3 \rangle$. Find $\mathbf{r}'(t)$. Sketch the curve, with an arrow to indicate the direction in which t increases. In your sketch, identify the position vector $\mathbf{r}(1)$ and draw the tangent vector $\mathbf{r}'(1)$.

Solution. We have $\mathbf{r}'(t) = \langle -2t, \ -3t^2 \rangle$, so $\mathbf{r}'(1) = \langle -2, \ -3 \rangle$. Also $\mathbf{r}(1) = \langle -1, \ -1 \rangle$.

By setting $t = -y^{1/3}$, we see that the points $(-y^{2/3}, \ y)$ are on the curve for all real numbers y, and that nothing else is. This gives the following graph:



The small circle is at $(-1,-1) = \mathbf{r}(1)$. The tangent vector $\mathbf{r}'(1)$ is drawn with starting at $\mathbf{r}(1)$, so that one can see that it is tangent to the curve. The arrows on the curve give the direction of increasing t. $\square$