MATH 281 (FALL 2020, PHILLIPS): COMMENTS ON AND SOLUTIONS TO HOMEWORK 4

1. Comments on mistakes

Common mistakes on Problem 1:

- Not doing what was asked for. Read the problem!
- Work not shown. I expect to see the equations of the traces you are supposed to graph.
- The trace in the plane $y = \sqrt{3}$ is *not* the trace in the xz-plane.
- Algebra mistakes. For example, the trace in the yz plane is *not* given by

$$\frac{z^2}{16} - \frac{y^2}{4} = 1.$$

- Use standard orientations of axes: z vertical if it is present; if only the variables x and y are present, then y is vertical.
- Label the axes and give scales. Otherwise, I can't tell if your graphs are right. (One person put tick marks but no scale. So is a tick mark one unit, 10 units, 0.01 units, or something else?)

Common mistakes on Problem 2:

- Various differentiation mistakes, especially when finding $\mathbf{r}''(t)$, and algebra mistakes.
- Despite its occasional appearance in the book, it is incorrect notation to write division of a vector by a scalar. One does not write

$$\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

  It must be

$$\left(\frac{1}{\|\mathbf{r}'(t)\|} \right) \mathbf{r}'(t) \qquad \text{or at least} \qquad \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t).$$

Common mistakes on Problem 3:

- Failure to simplify. For example, you must simplify $(0)(x)$ to 0 (and $(0)(x) + 18$ to 18), $(1)(x)$ to x,

$$\frac{-\cos^2(t^2) - \sin^2(t^2)}{2} \qquad \text{and} \qquad \frac{t\sin(t^2)}{t}$$

Date: 30 October 2020.

to

$$-\frac{1}{2} \qquad \text{and} \qquad \sin(t^2),$$

etc. Among other things, the problem deliberately specified $t > 0$ to enable the simplification $\sqrt{t^2} = t$. Without this, a correct solution is considerably more complicated.

- Many algebra mistakes.

2. Solutions

This homework assignment is due Friday 30 Oct. 2020 at 10:00 pm, to be uploaded as a pdf file (or one of a few other allowed file types) on the University of Oregon Canvas site.

General instructions: show work in all problems, and be very careful to use fully correct notation. Incorrect notation will lose credit on exams (grading is based on what you write, not what you meant), and the written homework assignments are your chance to have me tell you whether your notation is correct.

Files turned in must have good enough resolution that I can read them easily.

Apart from the extension (such as "`.pdf`"), your file name should contain only numbers, capital and lowercase letters, and underscores. In particular, **no** spaces or parentheses. You do not need to put any identifying information in the file name; "`HW4.pdf`" is quite sufficient. (The Canvas system adds enough identifying information.)

Write your name on all pages.

Problem 1 (8 points)**.** For the surface

$$\frac{z^2}{16} - \frac{(x-2)^2}{9} - \frac{y^2}{4} = 1,$$

find and draw the trace in the xy plane, the trace in the plane $y = \sqrt{3}$, and the trace in the plane $x = 0$. In your graphs, be sure to label the axes and put scales on the axes. Traces are in planes, so draw just the graph in a plane; don't try to locate it in space in your picture.

Solution. Although it isn't asked for in the problem, it is helpful to recognize that the surface is a two sheeted hyperboloid which opens up and down (along the vertical line through the point $(2, 0, 0)$), has center at $(2, 0, 0)$, and has vertices at $(2, 0, 4)$ and $(2,\ 0,\ -4)$.

The trace in the xy plane is gotten by setting $z = 0$:

$$-\frac{(x-2)^2}{9} - \frac{y^2}{4} = 1,$$

which has no solutions. The trace is the empty set. (You can just say this and draw nothing; or you can say this and draw axes without a graph. But if you draw axes without a graph, you need to say that the graph is empty, so that I don't think the picture just wasn't drawn.)

The trace in the plane $y = \sqrt{3}$ is

$$\frac{z^2}{16} - \frac{(x-2)^2}{9} - \frac{\left(\sqrt{3}\right)^2}{4} = 1,$$

which is

$$\frac{z^2}{16} - \frac{(x-2)^2}{9} = \frac{7}{4},$$

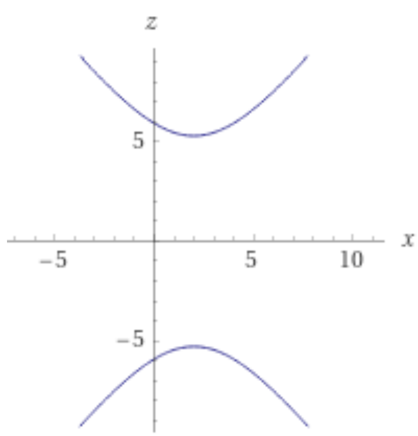
or

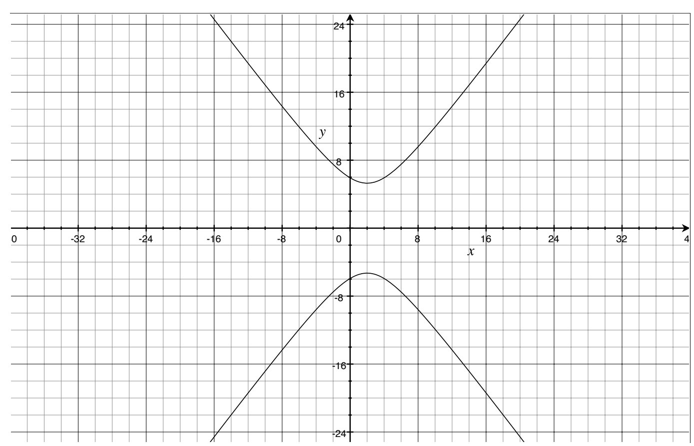
$$\frac{z^2}{(2\sqrt{7})^2} - \frac{(x-2)^2}{(3\sqrt{7}/2)^2} = 1.$$

It is a hyperbola with asymptotes

$$z = \pm\frac{4}{3}(x-2)$$

and vertices at $\left(2, 2\sqrt{7}\right)$ and $\left(2, -2\sqrt{7}\right)$. For the purposes of graphing, $2\sqrt{7} \approx 5.2195$. Here are graphs:





The one on the right shows more, but the vertical axis is mislabelled: it is the z-axis.

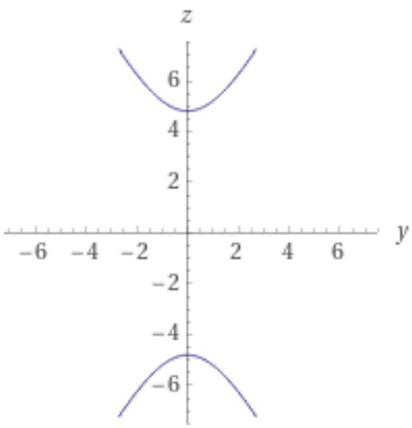
The trace in the plane $x = 0$ is

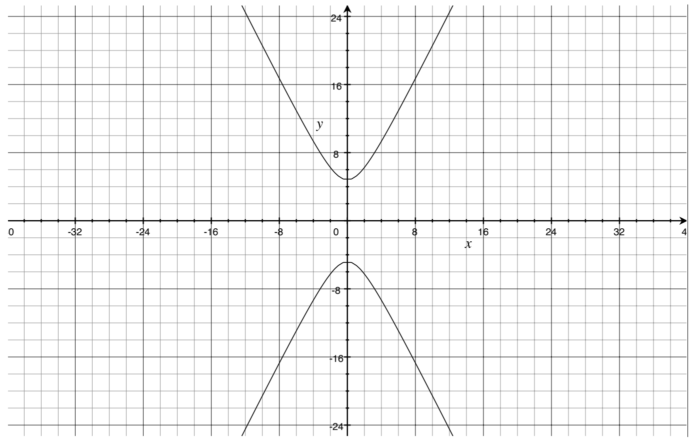
$$\frac{z^2}{16} - \frac{(-2)^2}{9} - \frac{y^2}{4} = 1,$$

which is

$$\frac{z^2}{(4\sqrt{13}/3)^2} - \frac{y^2}{(2\sqrt{13}/3)^2} = 1.$$

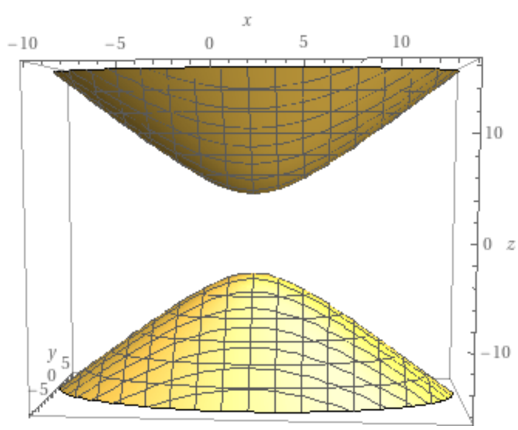
It is a hyperbola with asymptotes $z = \pm 2y$ and vertices at $(0,\, 2\sqrt{13}/3)$ and $(0,\, -2\sqrt{13}/3)$. For the purposes of graphing, $2\sqrt{13}/3 \approx 4.8074$. Here are graphs:





The one on the right shows more, but the axes are mislabelled: the horizontal axis is the y-axis the vertical axis is the z-axis. $\square$

The graph of the surface was *not* asked for, but here it is anyway:



Problem 2 (7 points)**.** Let f be a real valued function on $\mathbb{R}$ such that $f(2) = 2$, $f'(2) = -3$, and $f''(2) = 1$. Let $t \mapsto \mathbf{r}(t)$ be the curve in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = \big\langle f(t),\, t f(t),\, t^2 - 4t \big\rangle.$$

Find $\mathbf{r}'(t)$ (your answer will involve f and its derivatives), $\mathbf{T}(2)$, and the curvature $\kappa(2)$.

Solution. We differentiate coordinatewise, using the product rule on the second coordinate:

$$\mathbf{r}'(t) = \big\langle f'(t),\, f(t) + tf'(t),\, 2t - 4\big\rangle.$$

Therefore

$$\mathbf{r}'(2) = \langle -3,\, 2 + (2)(-3),\, 0\rangle = \langle -3,\, -4,\, 0\rangle,$$

$$\|\mathbf{r}'(2)\| = \sqrt{(-3)^2 + (-4)^2 + (0)^2} = 5,$$

and

$$\mathbf{T}(2) = \left(\frac{1}{\|\mathbf{r}'(2)\|}\right)\mathbf{r}'(2) = \left\langle -\frac{3}{5},\, -\frac{4}{5},\, 0\right\rangle.$$

For the curvature, we use the formula

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

So we need

$$\mathbf{r}''(t) = \big\langle f''(t),\, 2f'(t) + tf''(t),\, 2\big\rangle,$$

which gives

$$\mathbf{r}''(2) = \langle 1,\, 2(-3) + 2,\, 2\rangle = \langle 1,\, -4,\, 2\rangle.$$

Now

$$\begin{aligned}
&\mathbf{r}'(2) \times \mathbf{r}''(2) \\
&= \langle -3,\, -4,\, 0\rangle \times \langle 1,\, -4,\, 2\rangle \\
&= \big\langle (-4)(2) - (0)(-4),\, (0)(1) - (-3)(2),\, (-3)(-4) - (-4)(1)\big\rangle \\
&= \langle -8,\, 6,\, 16\rangle = 2\langle -4,\, -3,\, 8\rangle,
\end{aligned}$$

so

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 2\sqrt{(-4)^2 + (-3)^2 + (8)^2} = 2\sqrt{89},$$

and

$$\kappa(2) = \frac{2\sqrt{89}}{5^3} = \frac{2\sqrt{89}}{125}.$$

This completes the solution. □

Problem 3 (10 points)**.** Let $t \mapsto \mathbf{r}(t)$ be the curve in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = \big\langle \cos(t^2),\, \sqrt{3}t^2,\, \sin(t^2)\big\rangle,$$

for $t > 0$. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$. Further find equations of the normal and osculating planes at the point where $t = \sqrt{2\pi}$.

Solution. We have

$$\mathbf{r}'(t) = \left\langle -2t\sin(t^2),\ 2\sqrt{3}t,\ 2t\cos(t^2) \right\rangle.$$

Therefore, when $t > 0$ (so $\sqrt{t^2} = t$),

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(-2t\sin(t^2))^2 + \left(2\sqrt{3}t\right)^2 + (2t\cos(t^2))^2} \\ &= \sqrt{4t^2\left(\sin^2(t^2) + 3 + \cos^2(t^2)\right)} = 4t.\end{aligned}$$

So

$$\mathbf{T}(t) = \left(\frac{1}{\|\mathbf{r}'(t)\|}\right)\mathbf{r}'(t) = \left\langle -\frac{\sin(t^2)}{2},\ \frac{\sqrt{3}}{2},\ \frac{\cos(t^2)}{2} \right\rangle$$

and

$$\mathbf{T}'(t) = \left\langle -t\cos(t^2),\ 0,\ -t\sin(t^2) \right\rangle.$$

Therefore, when $t > 0$ (so $\sqrt{t^2} = t$),

$$\begin{aligned}\|\mathbf{T}'(t)\| &= \sqrt{[-t\cos(t^2)]^2 + [-t\sin(t^2)]^2} \\ &= \sqrt{t^2[\cos^2(t^2) + \sin^2(t^2)]} = \sqrt{t^2} = t.\end{aligned}$$

Thus

$$\mathbf{N}(t) = \left(\frac{1}{\|\mathbf{T}'(t)\|}\right)\mathbf{T}'(t) = \left\langle -\cos(t^2),\ 0,\ -\sin(t^2) \right\rangle.$$

So

$$\begin{aligned}\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \Bigg\langle \left(\frac{\sqrt{3}}{2}\right)(-\sin(t^2)) - \left(\frac{\cos(t^2)}{2}\right)(0), \\ &\qquad \left(\frac{\cos(t^2)}{2}\right)(-\cos(t^2)) - \left(-\frac{\sin(t^2)}{2}\right)(-\sin(t^2)), \\ &\qquad \left(-\frac{\sin(t^2)}{2}\right)(0) - \left(\frac{\sqrt{3}}{2}\right)(-\cos(t^2)) \Bigg\rangle \\ &= \left\langle -\left(\frac{\sqrt{3}}{2}\right)\sin(t^2),\ -\frac{1}{2},\ \left(\frac{\sqrt{3}}{2}\right)\cos(t^2) \right\rangle.\end{aligned}$$

Both requested planes go through the point $\mathbf{r}\left(\sqrt{2\pi}\right) = \left\langle 1,\ 2\pi\sqrt{3},\ 0 \right\rangle$. For the normal plane, we can take $\mathbf{r}'\left(\sqrt{2\pi}\right)$ or $\mathbf{T}\left(\sqrt{2\pi}\right)$ as the normal vector; I use

$$\mathbf{T}\left(\sqrt{2\pi}\right) = \left\langle 0,\ \frac{\sqrt{3}}{2},\ \frac{1}{2} \right\rangle$$

since it is easier to work with. This gives the equation

$$0(x-1)+\frac{\sqrt{3}}{2}\big(y-2\pi\sqrt{3}\big)+\frac{1}{2}z=0,$$

which we simplify and multiply by 2 to get

$$\sqrt{3}y+z=6\pi.$$

For the normal plane, we take $\mathbf{B}\big(\sqrt{2\pi}\big)$ as the normal vector. This gives the equation

$$0(x-1)-\frac{1}{2}\big(y-2\pi\sqrt{3}\big)+\frac{\sqrt{3}}{2}z=0,$$

which we simplify and multiply by -2 to get

$$y-\sqrt{3}z=2\sqrt{3}\pi.$$

This completes the solution. $\square$