MATH 281 (FALL 2020, PHILLIPS): COMMENTS ON AND SOLUTIONS TO HOMEWORK 6

1. Comments on mistakes

1.1. **Miscellaneous.** The partial derivative of, say, $z(x, y) = x^3 y - arctan(xe^{xy^5})$ with respect to x, is written

$$\frac{\partial z}{\partial x}, \qquad \text{not} \qquad \frac{d\not{z}}{\not{d}x}, \qquad \text{and not} \qquad \frac{\delta\not{z}}{\not{\delta}x}.$$

For single variable differentiation, if, say, $y(x) = -\sin^{11}(x + 3x^2)$, one writes

$$\frac{dy}{dx} \quad \text{or} \quad y'(x), \quad \text{not} \quad \frac{\partial\not{y}}{\not{\partial}x}, \quad \text{not} \quad \frac{\delta\not{y}}{\not{\delta}x}, \quad \text{and not} \quad \cancel{y_x}.$$

Use correct notation for differentiation operators. For example,

$$\frac{\partial}{\partial x} xyh(z) = \left(\frac{\partial}{\partial x}(x) \right) yh(z) = yh(z) \frac{\partial}{\partial x}(x),$$

which is not what was intended. In the expression

$$\frac{\partial}{\partial x}\big(xyh(z)\big),$$

the parentheses are *essential.*

If $f(x, y, z) = xyh(z) + \arctan(x^2 + 3z)$, then

$$\frac{\partial f}{\partial x} xyh(z)$$

means the product of the partial derivative $\frac{\partial f}{\partial x}(x, y, z)$ and the expression $xyh(z)$. So it is

$$\frac{\partial f}{\partial x} xyh(z) = yh(z)xyh(z) = xy^2 h(z)^2 \neq yh(z) = \frac{\partial}{\partial x}\big(xyh(z)\big).$$

In the other direction, it is wrong to write

$$\cancel{\frac{\partial}{\partial x} = yh(z)}.$$

One must take the partial derivative *of something*.

Date: 13 November 2020.

1.2. **Problem 1.** The derivative of h evaluated at z is $h'(z) = e^{-z^3} - z$. It is **not** true that ~~$h'(z) = e^{-t^3}$~~ $- t$. The right hand side is $h'(t)$, not $h'(z)$.

There is no need to multiply out $(x^2 + 3z)^2$. In context, this is not a simplification.

1.3. **Problem 2.** Since the hypotheses of Clairaut's Theorem include continuity of g_{xy} and g_{yx}, the fact that they are continuous *must* be mentioned in your reason.

It doesn't help to find g_{xx} or g_{yy}. They have nothing to do with a correct solution.

1.4. **Problem 3.** Use the chain rule!

1.5. **Problem 4.** Reminder on correct use of "=" and "≈": the sequence of computations

$$\begin{aligned} &f(3.02,\ -1.99,\ -1.005) \\ &\qquad \approx L(3.02,\ -1.99,\ -1.005) \\ &\qquad = -8 - 2(3.02 - 3) - 3(-1.99 + 2) - 4(-1.005 + 1) \\ &\qquad = -8.05 \end{aligned}$$

is correctly written. It stands for the following sequence of relations:

$$f(3.02,\ -1.99,\ -1.005) \approx L(3.02,\ -1.99,\ -1.005),$$

$$\begin{aligned} &L(3.02,\ -1.99,\ -1.005) \\ &\qquad = -8 - 2(3.02 - 3) - 3(-1.99 + 2) - 4(-1.005 + 1), \end{aligned}$$

and

$$-8 - 2(3.02 - 3) - 3(-1.99 + 2) - 4(-1.005 + 1) = -8.05.$$

In the second and third of these, using "≈" doesn't give a formally false statement, but doesn't show your work correctly and is therefore wrong in context.

Don't forget the constant term in the linear approximation!

It is true that the linear approximation can be rewritten as

$$L(x, y, z) = -2x - 3y - 4z - 12.$$

Doing this, however, makes it harder to evaluate $L(3.02,\ -1.99,\ -1.005)$, so it is a waste of time—time which, on an exam, could be spent working on another problem.

Always write, for example, 0.005, not .005. In the second version, especially with expressions like .46, the decimal point is too easily overlooked. People who actually work with numbers, such as physicists, know this. Unfortunately, people who write elementary calculus textbooks don't. (I won't take off points for ".005" or ".46", unless I actually misread them, but you should know not to write these expressions.)

2. Solutions

Problem 1 (9 points)**.** Let h be a function of one real variable such that $h'(t) = e^{-t^3} - t$ for all real numbers t. Let f be the function of three variables given by

$$f(x, y, z) = xyh(z) + \arctan(x^2 + 3z)$$

for all real numbers x, y, and z. Find all first order partial derivatives of f. Your answers may involve the function h.

Solution. We have:

$$\begin{aligned} f_x(x, y, z) &= \frac{\partial}{\partial x}\big(xyh(z)\big) + \frac{\partial}{\partial x}\big(\arctan(x^2 + 3z)\big) \\ &= yh(z)\frac{\partial}{\partial x}\big(x\big) + \frac{1}{1 + (x^2 + 3z)^2}\frac{\partial}{\partial x}\big(x^2 + 3z\big) \\ &= yh(z) + \frac{2x}{1 + (x^2 + 3z)^2}, \\ f_y(x, y, z) &= \frac{\partial}{\partial y}\big(xyh(z)\big) + \frac{\partial}{\partial y}\big(\arctan(x^2 + 3z)\big) \\ &= xh(z)\frac{\partial}{\partial y}\big(y\big) + 0 = xh(z), \end{aligned}$$

and

$$\begin{aligned} f_z(x, y, z) &= \frac{\partial}{\partial z}\big(xyh(z)\big) + \frac{\partial}{\partial z}\big(\arctan(x^2 + 3z)\big) \\ &= xy\frac{d}{dz}\big(h(z)\big) + \frac{1}{1 + (x^2 + 3z)^2}\frac{\partial}{\partial z}\big(x^2 + 3z\big) \\ &= xyh'(z) + \frac{3}{1 + (x^2 + 3z)^2} = xy(e^{-z^3} - z) + \frac{3}{1 + (x^2 + 3z)^2}. \end{aligned}$$

This is all that was asked for. □

Problem 2 (4 points)**.** Explain why there is no function g of two real variables such that $g_x(x, y) = 3x^2 + y^3$ and $g_y(x, y) = x^3 + 3xy^2$ for all real numbers x and y.

Hint: consider what the second order partial derivatives would need to be.

Note: showing correct work is important.

Solution. Suppose there were such a function g. Computing partial derivatives would give

$$g_{xy}(x,y) = \frac{\partial}{\partial y}\big(3x^2 + y^3\big) = 3y^2$$

and

$$g_{yx}(x,y) = \frac{\partial}{\partial x}\big(x^3 + 3xy^2\big) = 3x^2 + 3y^2.$$

These are both continuous. By a theorem in the book, they must therefore be equal. But they disagree at $(1,0)$, since the first expression has the value 0 there while the second expression has the value 3 there. □

(The theorem is Clairuat's Theorem, on page 959.)

Problem 3 (6 points)**.** Let h be a function of three variables such that

$$h_x(x,y) = -2xy + \sin(e^x) \qquad \text{and} \qquad h_y(x,y) = -x^2$$

for all real numbers x and y. Define a function z by $z(t) = h(t^3,\, t\cos(t))$ for all real numbers t. Find $z'(t)$. Remember to simplify your answer!

Solution. We use the chain rule:

$$\begin{aligned} z'(t) &= h_x(t^3,\, t\cos(t))\frac{d}{dt}\big(t^3\big) + h_y(t^3,\, t\cos(t))\frac{d}{dt}\big(t\cos(t)\big) \\ &= \big[-2t^3(t\cos(t)) + \sin\big(e^{t^3}\big)\big](3t^2) - (t^3)^2\big(\cos(t) - t\sin(t)\big) \\ &= 3t^2\big[-2t^4\cos(t) + \sin\big(e^{t^3}\big)\big] - t^6\big(\cos(t) - t\sin(t)\big). \\ &= 3t^2\sin\big(e^{t^3}\big) - 7t^6\cos(t) + t^7\sin(t). \end{aligned}$$

□

Problem 4 (6 points)**.** Suppose you have a function f of three real variables which is differentiable at $(3,\,-2,\,-1)$ and such that

$$f(3,\,-2,\,-1) = -8, \quad f(0,0,0) = -11,$$

$$f_x(3,\,-2,\,-1) = -2, \quad f_y(3,\,-2,\,-1) = -3, \quad f_z(3,\,-2,\,-1) = -4,$$

$$f_x(3,0,0) = -6, \quad f_y(0,\,-2,\,0) = 3, \quad \text{and} \quad f_z(0,\,0,\,-1) = 7.$$

Use the linear approximation at $(3,\,-2,\,-1)$ to estimate

$$f(3.02,\,-1.99,\,-1.005),$$

or explain why there is not enough information to do so.

You will not need to use all the information provided.

Be sure to keep the symbols "=" and "≈" straight: use each of them where mathematically appropriate, but don't use one where only the other is correct.

Solution. Let $L(x, y, z)$ be the linear approximation to $f(x, y, z)$ based at the point $(3, -2, -1)$. Then

$$\begin{aligned} L(x, y, z) &= f(3, -2, -1) + f_x(3, -2, -1)(x - 3) \\ &\qquad + f_y(3, -2, -1)(y - (-2)) + f_z(3, -2, -1)(z - (-1)) \\ &= -8 - 2(x - 3) - 3(y + 2) - 4(z + 1), \end{aligned}$$

so

$$\begin{aligned} &f(3.02, -1.99, -1.005) \\ &\qquad \approx L(3.02, -1.99, -1.005) \\ &\qquad = -8 - 2(3.02 - 3) - 3(-1.99 + 2) - 4(-1.005 + 1) \\ &\qquad = -8.05. \end{aligned}$$

Look carefully where one writes "=" and where one writes "≈". □