MATH 281 (FALL 2020, PHILLIPS): SOLUTIONS TO HOMEWORK 7

Problem 1 (11 points). Set $g(x, y) = x^2 + 4xy - 2x + 2y^4 - 4y + 1$. Find and classify the critical points of g. Be sure that your work shows that you have found all of them.

Solution. We have

$$D_1 g(x, y) = 2x + 4y - 2 \qquad \text{and} \qquad D_2 g(x, y) = 4x + 8y^3 - 4.$$

In particular, they exist everywhere. To find the critical points, we therefore solve the simultaneous equations

$$2x + 4y - 2 = 0 \qquad \text{and} \qquad 4x + 8y^3 - 4 = 0. \tag{1}$$

The first equation implies $x = 1 - 2y$. Substitute this in the second equation to get

$$4(1 - 2y) + 8y^3 - 4 = 0.$$

Simplify and rearrange:

$$0 = 8y^3 - 8y = 8y(y - 1)(y + 1).$$

Its solutions are $y = 0$, $y = 1$, and $y = -1$. If $y = 0$ then the first equation implies $x = 1$. If $y = 1$ then the first equation implies $x = -1$. If $y = -1$ then the first equation implies $x = 3$. The points $(1, 0)$, $(-1, 1)$, and $(3, -1)$ do in fact all satisfy (1), so these are exactly the critical points.

To find the types of the critical points, we need the second partial derivatives,

$$D_1^2(x, y) = 2, \qquad D_2^2 g(x, y) = 24y^2,$$

and

$$D_1 D_2 g(x, y) = D_2 D_1 g(x, y) = 4.$$

So, at the critical point (a, b),

$$D(a, b) = (2)(24b^2) - 4^2 = 48b^2 - 16.$$

Then $D(1, 0) = -16 < 0$, so g has a saddle point at $(1, 0)$. Also, $D(-1, 1) = 32 > 0$, and $D_1^2 g(-1, 1) = 2 > 0$, so g has a local minimum at $(-1, 1)$. Finally, $D(3, -1) = 32 > 0$, and $D_1^2 g(3, -1) = 2 > 0$, so g has a local minimum at $(3, -1)$. □

The equations for the critical points are simultaneous in x and y. When you find that $x = 1$ and $y = 0$ satisfy these equations, and that $x = -1$ and $y = 1$ satisfy these equations, this doesn't say that $x = 1$ and $y = 1$ satisfy these equations. So, in this problem, there is no need to consider the

Date: 8 December 2020.

point $(1, 1)$. Similarly, there is no need to consider any of $(1, -1)$, $(-1, 0)$, $(-1, -1)$, $(3, 0)$, or $(3, 1)$. (Of course, finding that $x = 1$ and $y = 0$ satisfy these equations, and also that $x = -1$ and $y = 1$ satisfy these equations, does not rule out the possibility that $x = 1$ and $y = 1$ satisfy these equations. It depends on the particular equations you have.)

Problem 2 (6 points)**.** Let f be a differentiable function of two variables. Let $g(r, s) = f\big(3r^2 s - 1,\, r^2 - s\cos(s)\big)$. Suppose

$$f(1, 0) = 3, \quad g(1, 0) = -7, \quad D_1 f(1, 0) = 4, \quad D_2 f(1, 0) = 8,$$

$$f(-1, 1) = -7, \quad g(-1, 1) = 3, \quad D_1 f(-1, 1) = 2, \quad D_2 f(-1, 1) = -5,$$

and

$$f(3, -1) = 11, \quad g(3, -1) = -2, \quad D_1 f(3, -1) = -9, \quad D_2 f(3, -1) = 7.$$

Find $D_2 g(1, 0)$.

Solution. Set

$$x(r, s) = 3r^2 s - 1 \qquad \text{and} \qquad y(r, s) = r^2 - s\cos(s).$$

Then

$$D_2 x(r, s) = 3r^2 \qquad \text{and} \qquad D_2 y(r, s) = -\cos(s) + s\sin(s).$$

So

$$D_2 x(1, 0) = 3 \qquad \text{and} \qquad D_2 y(1, 0) = -1.$$

Also

$$x(1, 0) = -1 \qquad \text{and} \qquad y(1, 0) = 1.$$

So

$$\begin{aligned} D_2 g(1, 0) &= D_1 f\big(x(1, 0),\, y(1, 0)\big) D_2 x(1, 0) + D_2 f\big(x(1, 0),\, y(1, 0)\big) D_2 y(1, 0) \\ &= D_1 f(-1, 1) D_2 x(1, 0) + D_2 f(-1, 1) D_2 y(1, 0) \\ &= (2)(3) + (-5)(-1) = 11. \end{aligned}$$

This completes the solution. □

Problem 3 (7 points)**.** For $(x, y) \neq (0, 0)$, define

$$h_1(x, y) = -\frac{y}{x^2 + y^2} \qquad \text{and} \qquad h_2(x, y) = \frac{x}{x^2 + y^2}.$$

Show that there is no function f defined for all points (x, y) in $\mathbb{R}^2$ with $(x, y) \neq (0, 0)$ whose first and second order partial derivatives all exist and are continuous on its domain and such that $D_1(x, y)f = h_1(x, y)$ and $D_2(x, y)f = h_2(x, y)$ for all points (x, y) in its domain. Explain why this does not violate Clairaut's Theorem, on page 959 of the book.

Hint. Suppose such a function f exists. Look at what the derivative of the function $q(t) = f(\cos(t), \sin(t))$ would have to be. Use this information to compare $q(2\pi)$ with $q(0)$, and find something wrong with the result you get.

Solution. We first look for an easy solution. Since the two mixed second order partial derivatives of f are supposed to be equal, we compare $D_2h_1(x, y)$ (which would be $D_2D_1f(x, y)$) and $D_1h_2(x, y)$ (which would be $D_1D_2f(x, y)$). These are computed via the quotient rule: whenever $(x, y) \neq (0, 0)$, we have

$$D_2h_1(x, y) = -\frac{(1)(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$D_1h_2(x, y) = \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = D_2h_1(x, y).$$

Therefore Clairaut's Theorem is of no help proving that f does not exist.

Suppose such a function f exists. Following the hint, define $q(t) = f(\cos(t), \sin(t))$ for all real numbers t. Differentiate using the chain rule:

$$\begin{aligned} q'(t) &= D_1f(\cos(t), \sin(t))\cos'(t) + D_2f(\cos(t), \sin(t))\sin'(t) \\ &= \left(-\frac{\sin(t)}{\cos^2(t) + \sin^2(t)}\right)(-\sin(t)) + \left(\frac{\cos(t)}{\cos^2(t) + \sin^2(t)}\right)\cos(t) \\ &= \frac{\sin^2(t) + \cos^2(t)}{\cos^2(t) + \sin^2(t)} = 1. \end{aligned}$$

Therefore there is a constant C such that $q(t) = t + C$. Now

$$\begin{aligned} 2\pi + C = q(2\pi) &= f(\cos(2\pi), \sin(2\pi)) = f(1, 0) \\ &= f(\cos(0), \sin(0)) = q(0) = C. \end{aligned}$$

This is obviously impossible, so no such function f exists.

To see why this does not contradict Clairaut's Theorem, rewrite Clairaut's Theorem as follows. Assume we are given functions h_1 and h_2 of two variables, both defined on a suitable (that is, open) set D, such that $D_2h_1(x, y)$ and $D_1h_2(x, y)$ exist and are continuous on D. Clairaut's Theorem states that if there is a function f such that $h_1(x, y) = D_1f(x, y)$ and $h_2(x, y) = D_2f(x, y)$ for all points (x, y) in D, then $D_2h_1(x, y) = D_1h_2(x, y)$ for all points (x, y) in D.

Clairaut's Theorem does *not* say that if $D_2h_1(x, y) = D_1h_2(x, y)$ for all points (x, y) in D, then there is a function f such that $h_1(x, y) = D_1f(x, y)$ and $h_2(x, y) = D_2f(x, y)$ for all points (x, y) in D. As the example in the problem shows, this statement is false. □

Integrating $h_1(x, y)$ with respect to x gives $-\arctan(x/y)$ plus a constant depending on y, and integrating $h_2(x, y)$ with respect to y gives $\arctan(y/x)$ plus a constant depending on x. This does *not* lead to an easy contradiction, because

$$\arctan\left(\frac{1}{t}\right) = \frac{\pi}{2} - \arctan(t)$$

if $t > 0$, and

$$\arctan\left(\frac{1}{t}\right) = -\frac{\pi}{2} - \arctan(t)$$

if $t < 0$. (The easiest way to check this is to show that the derivative of $\arctan(t) + \arctan\left(\frac{1}{t}\right)$ is zero, and then evaluate this function at $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.)