Problem 1 (11 points). Set \( g(x, y) = x^2 + 4xy - 2x + 2y^4 - 4y + 1 \). Find and classify the critical points of \( g \). Be sure that your work shows that you have found all of them.

Solution. We have

\[
D_1 g(x, y) = 2x + 4y - 2 \quad \text{and} \quad D_2 g(x, y) = 4x + 8y^3 - 4.
\]

In particular, they exist everywhere. To find the critical points, we therefore solve the simultaneous equations

\[
(1) \quad 2x + 4y - 2 = 0 \quad \text{and} \quad 4x + 8y^3 - 4 = 0.
\]

The first equation implies \( x = 1 - 2y \). Substitute this in the second equation to get

\[
4(1 - 2y) + 8y^3 - 4 = 0.
\]

Simplify and rearrange:

\[
0 = 8y^3 - 8y = 8y(y - 1)(y + 1).
\]

Its solutions are \( y = 0 \), \( y = 1 \), and \( y = -1 \). If \( y = 0 \) then the first equation implies \( x = 1 \). If \( y = 1 \) then the first equation implies \( x = -1 \). If \( y = -1 \) then the first equation implies \( x = 3 \). The points \((1, 0)\), \((-1, 1)\), and \((3, -1)\) do in fact all satisfy (1), so these are exactly the critical points.

To find the types of the critical points, we need the second partial derivatives,

\[
D_1^2 g(x, y) = 2, \quad D_2^2 g(x, y) = 24y^2,
\]

and

\[
D_1 D_2 g(x, y) = D_2 D_1 g(x, y) = 4.
\]

So, at the critical point \((a, b)\),

\[
D(a, b) = (2)(24b^2) - 4^2 = 48b^2 - 16.
\]

Then \( D(1, 0) = -16 < 0 \), so \( g \) has a saddle point at \((1, 0)\). Also, \( D(-1, 1) = 32 > 0 \), and \( D_1^2 g(-1, 1) = 2 > 0 \), so \( g \) has a local minimum at \((-1, 1)\). Finally, \( D(3, -1) = 32 > 0 \), and \( D_1^2 g(3, -1) = 2 > 0 \), so \( g \) has a local minimum at \((3, -1)\).

The equations for the critical points are simultaneous in \( x \) and \( y \). When you find that \( x = 1 \) and \( y = 0 \) satisfy these equations, and that \( x = -1 \) and \( y = 1 \) satisfy these equations, this doesn’t say that \( x = 1 \) and \( y = 1 \) satisfy these equations. So, in this problem, there is no need to consider the

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point (1, 1). Similarly, there is no need to consider any of (1, −1), (−1, 0), (−1, −1), (3, 0), or (3, 1). (Of course, finding that $x = 1$ and $y = 0$ satisfy these equations, and also that $x = −1$ and $y = 1$ satisfy these equations, does not rule out the possibility that $x = 1$ and $y = 1$ satisfy these equations. It depends on the particular equations you have.)

Problem 2 (6 points). Let $f$ be a differentiable function of two variables. Let $g(r, s) = f(3r^2s − 1, r^2 − s \cos(s))$. Suppose

\[
\begin{align*}
f(1, 0) &= 3, & g(1, 0) &= −7, & D_1f(1, 0) &= 4, & D_2f(1, 0) &= 8, \\
f(−1, 1) &= −7, & g(−1, 1) &= 3, & D_1f(−1, 1) &= 2, & D_2f(−1, 1) &= −5, \\
f(3, −1) &= 11, & g(3, −1) &= −2, & D_1f(3, −1) &= −9, & D_2f(3, −1) &= 7.
\end{align*}
\]

Find $D_2g(1, 0)$.

Solution. Set $x(r, s) = 3r^2s − 1$ and $y(r, s) = r^2 − s \cos(s)$. Then

\[
\begin{align*}
D_2x(r, s) &= 3r^2 & \text{and} & D_2y(r, s) &= −\cos(s) + s \sin(s).
\end{align*}
\]

So

\[
\begin{align*}
D_2x(1, 0) &= 3 & \text{and} & D_2y(1, 0) &= −1.
\end{align*}
\]

Also

\[
\begin{align*}
x(1, 0) &= −1 & \text{and} & y(1, 0) &= 1.
\end{align*}
\]

So

\[
\begin{align*}
D_2g(1, 0) &= D_1f(x(1, 0), y(1, 0))D_2x(1, 0) + D_2f(x(1, 0), y(1, 0))D_2y(1, 0) \\
&= D_1f(−1, 1)D_2x(1, 0) + D_2f(−1, 1)D_2y(1, 0) \\
&= (2)(3) + (−5)(−1) = 11.
\end{align*}
\]

This completes the solution. \qed

Problem 3 (7 points). For $(x, y) \neq (0, 0)$, define

\[
\begin{align*}
h_1(x, y) &= −\frac{y}{x^2 + y^2} & \text{and} & h_2(x, y) &= \frac{x}{x^2 + y^2}.
\end{align*}
\]

Show that there is no function $f$ defined for all points $(x, y)$ in $\mathbb{R}^2$ with $(x, y) \neq (0, 0)$ whose first and second order partial derivatives all exist and are continuous on its domain and such that $D_1(x, y)f = h_1(x, y)$ and $D_2(x, y)f = h_2(x, y)$ for all points $(x, y)$ in its domain. Explain why this does not violate Clairaut’s Theorem, on page 959 of the book.

Hint. Suppose such a function $f$ exists. Look at what the derivative of the function $q(t) = f(\cos(t), \sin(t))$ would have to be. Use this information to compare $q(2\pi)$ with $q(0)$, and find something wrong with the result you get.
Therefore there is a constant $C$ and $f$ of second order partial derivatives of $f$ are supposed to be equal, we compare $D_2 h_1(x, y)$ (which would be $D_2 D_1 f(x, y)$) and $D_1 h_2(x, y)$ (which would be $D_1 D_2 f(x, y)$). These are computed via the quotient rule: whenever $(x, y) \neq (0, 0)$, we have

$$D_2 h_1(x, y) = -\frac{(1)(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$D_1 h_2(x, y) = \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = D_2 h_1(x, y).$$

Therefore Clairaut’s Theorem is of no help proving that $f$ does not exist.

Suppose such a function $f$ exists. Following the hint, define $q(t) = f(\cos(t), \sin(t))$ for all real numbers $t$. Differentiate using the chain rule:

$$q'(t) = D_1 f(\cos(t), \sin(t)) \cos'(t) + D_2 f(\cos(t), \sin(t)) \sin'(t)$$

$$= \left( -\frac{\sin(t)}{\cos^2(t) + \sin^2(t)} \right)(-\sin(t)) + \left( \frac{\cos(t)}{\cos^2(t) + \sin^2(t)} \right) \cos(t)$$

$$= \frac{\sin^2(t) + \cos^2(t)}{\cos^2(t) + \sin^2(t)} = 1.$$ 

Therefore there is a constant $C$ such that $q(t) = t + C$. Now

$$2\pi + C = q(2\pi) = f(\cos(2\pi), \sin(2\pi)) = f(1, 0)$$

$$= f(\cos(0), \sin(0)) = q(0) = C.$$ 

This is obviously impossible, so no such function $f$ exists.

To see why this does not contradict Clairaut’s Theorem, rewrite Clairaut’s Theorem as follows. Assume we are given functions $h_1$ and $h_2$ of two variables, both defined on a suitable (that is, open) set $D$, such that $D_2 h_1(x, y)$ and $D_1 h_2(x, y)$ exist and are continuous on $D$. Clairaut’s Theorem states that if there is a function $f$ such that $h_1(x, y) = D_1 f(x, y)$ and $h_2(x, y) = D_2 f(x, y)$ for all points $(x, y)$ in $D$, then $D_2 h_1(x, y) = D_1 h_2(x, y)$ for all points $(x, y)$ in $D$.

Clairaut’s Theorem does not say that if $D_2 h_1(x, y) = D_1 h_2(x, y)$ for all points $(x, y)$ in $D$, then there is a function $f$ such that $h_1(x, y) = D_1 f(x, y)$ and $h_2(x, y) = D_2 f(x, y)$ for all points $(x, y)$ in $D$. As the example in the problem shows, this statement is false. \(\square\)

Integrating $h_1(x, y)$ with respect to $x$ gives $-\arctan(x/y)$ plus a constant depending on $y$, and integrating $h_2(x, y)$ with respect to $y$ gives $\arctan(y/x)$ plus a constant depending on $x$. This does not lead to an easy contradiction, because

$$\arctan\left(\frac{1}{t}\right) = \frac{\pi}{2} - \arctan(t)$$
if $t > 0$, and
\[
\arctan\left(\frac{1}{t}\right) = -\frac{\pi}{2} - \arctan(t)
\]
if $t < 0$. (The easiest way to check this is to show that the derivative of $\arctan(t) + \arctan\left(\frac{1}{t}\right)$ is zero, and then evaluate this function at $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.)