MATH 281 (FALL 2020, PHILLIPS): SOLUTIONS TO HOMEWORK 8

Problem 1 (10 points)**.** Use Lagrange multipliers to find the maximum and minimum values of $4x + \frac{1}{2}y$ on the surface given by $x^4 = 17 - y^4$, and where they occur.

Solution. We set

$$f(x, y) = 4x + \frac{y}{2} \qquad \text{and} \qquad g(x, y) = x^4 + y^4.$$

The Lagrange multiplier equations are then

$$g(x, y) = 17, \quad D_1 f(x, y) = \lambda D_1 g(x, y), \quad \text{and} \quad D_2 f(x, y) = \lambda D_2 g(x, y).$$

Putting in what the functions are, we get

$$x^4 + y^4 = 17, \qquad 4 = \lambda(4x^3), \qquad \text{and} \qquad \frac{1}{2} = \lambda(4y^3). \tag{1}$$

The last two equations become

$$x^3 = \frac{1}{\lambda} \qquad \text{and} \qquad y^3 = \frac{1}{8\lambda}. \tag{2}$$

So

$$y = \frac{1}{2\lambda^{1/3}} \qquad \text{and} \qquad x = \frac{1}{\lambda^{1/3}} = 2y.$$

Therefore

$$17 = x^4 + y^4 = (2y)^4 + y^4 = 17y^4.$$

So $y = \pm 1$. For both $y = 1$ and $y = -1$, we get $17 = x^4 + 1$, so $x = \pm 2$. Thus, we need to compare

$$f(2, 1) = \frac{17}{2}, \quad f(2, -1) = \frac{15}{2}, \quad f(-2, 1) = -\frac{15}{2}, \quad \text{and} \quad f(-2, -1) = -\frac{17}{2}.$$

Therefore the maximum value is $\frac{17}{2}$, which occurs at $(2, 1)$, and the minimum value is $-\frac{17}{2}$, which occurs at $(-2, -1)$. $\square$

The points $(2, -1)$ and $(-2, 1)$ do not actually come from solutions of (1), since they are inconsistent with (2). They appeared in our list because we used the equation $x^4 + y^4 = 17$ to get x from y. There is no damage done (except for a possible waste of time) from evaluation f at more points than needed.

For grading purposes: if you use $x^4 + y^4 = 17$ to get x from y but give no indication that you recognize that doing so also produces the points $(2, -1)$ and $(-2, 1)$, that is an error. If instead you use (2) to get x from y, as

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for example in the alternate solution below, but then claim that the points $(2, -1)$ and $(-2, 1)$ are solutions, that is also an error.

Alternate solution. This solution differs only in the method used to solve the equations (1). From the last two equations in (1), we get

$$\frac{1}{\lambda} = x^3 \qquad \text{and} \qquad \frac{1}{\lambda} = 8y^3.$$

Therefore $x^3 = 8y^3$. Since we exclude complex numbers, this equation implies $x = 2y$. Therefore (as in the first solution)

$$17 = x^4 + y^4 = (2y)^4 + y^4 = 17y^4.$$

So $y = \pm 1$. If $y = 1$ then the equation $x = 2y$ implies $x = 2$; if $y = -1$ then the equation $x = 2y$ implies $x = -2$.

Now compare $f(2, 1) = \frac{17}{2}$ and $f(-2, -1) = -\frac{17}{2}$ as in the first solution, ignoring $f(-2, 1)$ and $f(2, -1)$. □

Problem 2 (10 points)**.** Find the maximum and minimum values of $x^2 + 2y - xy$ on the closed filled in triangle T with vertices at $(0, 0)$, $(0, 2)$, and $(4, 0)$, and where they occur.

Solution. Unfortunately, this solution is missing a picture, but we can describe T as the region in the first quadrant bounded by the x-axis, the y-axis, and the line $y = 2 - \frac{1}{2}t$. Alternatively,

$$T = \left\{ (x, y) \colon 0 \leq x \leq 4 \text{ and } 0 \leq y \leq 2 - \tfrac{1}{2}t \right\}.$$

Set $f(x, y) = x^2 + 2y - xy$. We begin by finding the critical points of f, which means solving the equations

$$0 = D_1 f(x, y) = 2x - y \qquad \text{and} \qquad 0 = D_2 f(x, y) = 2 - x.$$

These imply $x = 2$ and $y = 2x = 4$. So the only critical point is at $(2, 4)$. Since $(2, 4)$ is not in T, we ignore it. (I **must** see that you found this critical point and recognized that it is not in T.)

We then look for the maximum and minimum values on the parts of the boundary.

We parametrize the part along the x-axis by $t \mapsto (t, 0)$ for t in $[0, 4]$, leading to the function $g(t) = f(t, 0) = t^2$ on $[0, 4]$. Even without calculus, its minimum value is obviously 0 and its maximum value is obviously 16.

We parametrize the part along the y-axis by $t \mapsto (0, t)$ for t in $[0, 2]$, leading to the function $h(t) = f(0, t) = 2t$ on $[0, 2]$. Even without calculus, its minimum value is obviously 0 and its maximum value is obviously 4.

We parametrize the diagonal part by $t \mapsto \left(t, 2 - \frac{1}{2}t\right)$ for t in $[0, 4]$, leading to the function

$$k(t) = f\left(t, 2 - \frac{1}{2}t\right) = t^2 + 2\left(2 - \frac{t}{2}\right) - t\left(2 - \frac{t}{2}\right) = \frac{1}{2}(3t^2 - 6t + 8)$$

on $[0,4]$. To find its extreme values on $[0,4]$, we differentiate and set the derivative equal to zero:

$$0 = k'(t) = 3t - 3.$$

So we solve $3t - 3 = 0$, getting $t = 1$. We therefore compare the values

$$k(0) = 4, \qquad k(4) = 16, \qquad \text{and} \qquad k(1) = \frac{5}{2}.$$

The largest of all the values we got is 16, which occurred at $(4,0)$, and the smallest of all the values we got is 0, which occurred at $(0,0)$, $\square$

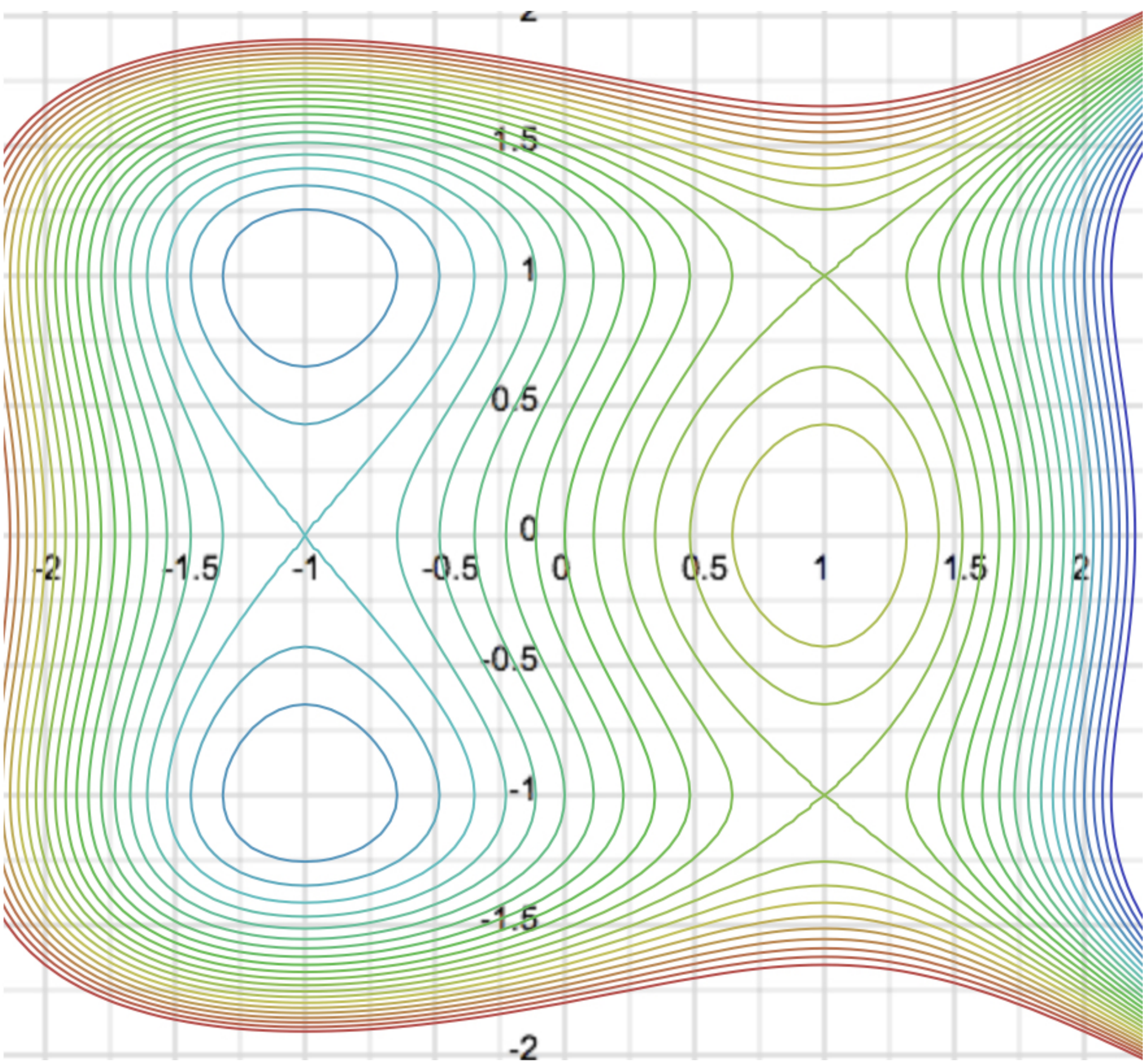
An alternate parametrization for the diagonal part is $t \mapsto (4 - 2t,\, t)$ on for t in $[0,2]$, which leads to the function

$$l(t) = (4 - 2t)^2 + 2t - (4 - 2t)t = 6t^2 - 18t + 16.$$

A correct solution must clearly demonstrate that you have considered possible extreme values *both* along the interiors of the edges and at their endpoints (which are the vertices of the triangle).

(Problem 3 on next page.)

Problem 3 (5 points)**.** The picture below is a partial of a contour plot of a function $z = Q(x, y)$. The contour lines are evenly spaced, with the darkest red (at the top and bottom) being at the value 4 and the darkest blue (at the right) being at the value -4.



For each of the following quantities, determine whether it is near zero, clearly positive, or clearly negative. In each case give a brief explanation.

(1) $D_2Q(0, 0)$.
(2) $D_1^2Q(-1, 1)$. (In the book's most common notation, this is $Q_{xx}(-1, 1)$.)
(3) $Q(0.5, 1.5) - Q(0.5, 2)$. (Assume the behavior of the function continues in the same way beyond the last contour line shown.)
(4) $D_1Q(0, -1.5)$.

(5) $D_2Q(0, -1.5)$.

Solution. For convenience, we think of this as a topographic map, and follow the usual map convention, in which right is east and up is north. Thus redder contours are higher and bluer contours are lower.

(1) $D_2Q(0,0) \approx 0$, since as you go north through $(0,0)$, you are going up (away from blue) before you get there, but down (towards blue) after you leave this point.

(2) $D_1^2Q(-1,1) > 0$, since as you go east through $(-1,1)$ you are going through the bottom of a valley (or hole): the contours get redder (higher) in both directions. (Note: the problem is *not* about $D_1Q(-1,1)$. Many reasons given seemed to be reasons for a value of $D_1Q(-1,1)$, usually connected with a wrong answer for this value. The quantity $D_1Q(-1,1)$ is near zero.)

(3) $Q(0.5,1.5)-Q(0.5,2) > 0$, since, as you go from $(0.5,1.5)$ to $(0.5,2)$, you are going up. (The contours get redder (higher), until you pass the last one shown; according to the instructions, you should assume they continue the same way.)

(4) $D_1Q(0, -1.5) > 0$, since as you go east through $(0, -1.5)$, you are going towards red, that is, up.

(5) $D_2Q(0, -1.5) < 0$, since as you go north through $(0, -1.5)$, you are going away from red, that is, down.

□