MATH 343 (SPRING 2021, PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 1

The statements of the problems are not identical to what is in the book.

Problem 1 (Problem 1.2 in the book)**.** You choose four mice at random, from a population of lab mice bred to have mass 30 gram, give or take 5 grams or so. Interpreting this as expressing the standard error, and using the Pythagorean property of standard errors, what is the total mass or your four mice, give or take how much?

Solution. The total mass will be the sum of the masses, $(4)(30) = 120$ grams, give or take the square root of the sum of the squares of the errors, $\sqrt{5^2+5^2+5^2+5^2} = \sqrt{100} = 10$ grams. □

Problem 2 (Problem 1.3 in the book)**.** Using the book's table of random numbers or a random number generator, simulate four times the choice of a sample of size 25 from a population with a probability $\pi = 0.6$ of a positive outcome. For your simulation, find:

(1) The average of the sample proportions p.
(2) The mean of the sampling errors.
(3) The mean of the absolute values of the sampling errors.
(4) The root mean square of the sampling errors.

Solution. Ansers will vary depending on the choice of random numbers.

The random number generator I used gave sample counts of 17, 21, 14, and 13, hence sample proportions of 0.68, 0.84, 0.56, and 0.52. The errors are therefore

$$0.68-0.6 = 0.08, \quad 0.84-0.6 = 0.24, \quad 0.56-0.6 = -0.04, \quad \text{and} \quad 0.52-0.6 = -0.08.$$

For (1), I therefore get

$$\frac{0.68+0.84+0.56+0.52}{4} = 0.65.$$

For (2), I get

$$\frac{0.08+0.24+(-0.04)+(-0.08)}{4} = 0.05.$$

For (3), I get

$$\frac{|0.08|+|0.24|+|-0.04|+|-0.08|}{4} = 0.11.$$

For (4), I get

$$\sqrt{\frac{(0.08)^2+(0.24)^2+(-0.04)^2+(-0.08)^2}{4}} = \sqrt{\frac{0.072}{4}} \approx 0.13416.$$

□

Problem 3 (Problem 1.7 in the book)**.** You plan to take a random phone survey on a yes or no question.

(1) Use the bound given in (1.1) in Section 1.3 of the book to estimate the sample size needed to get a standard error less than 5%.
(2) What should the sample size be to get a an error less than 5% with 95% confidence?

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(3) What should the sample size be to get a an error less than 5% with 99.7% confidence?

Solution. For (1), we want the standard error SE to satisfy SE ≤ 0.05. We therefore need

$$\frac{1}{2\sqrt{n}} \leq \frac{1}{20},$$

so $\sqrt{n} \geq 10$. Thus $n = 100$ will do.

For (2), we want the standard error SE to satisfy $2 \cdot \text{SE} \leq 0.05$. We therefore need

$$2\left(\frac{1}{2\sqrt{n}}\right) \leq \frac{1}{20},$$

so $\sqrt{n} \geq 20$. Thus $n = 400$ will do.

For (3), we want the standard error SE to satisfy $3 \cdot \text{SE} \leq 0.05$. We therefore need

$$3\left(\frac{1}{2\sqrt{n}}\right) \leq \frac{1}{20},$$

so $\sqrt{n} \geq 30$. Thus $n = 900$ will do. □

Problem 4 (Problem 1.13 in the book)**.** Use methods of calculus to show that the maximum value of the function $f(x) = x(1-x)$ occurs at $x = \frac{1}{2}$. Then use this to prove that

$$\sqrt{\frac{\pi(1-\pi)}{n}} \leq \frac{1}{2\sqrt{n}}$$

for any π in $[0, 1]$.

Solution. For the first part, we write $f(x) = x - x^2$, and calculate $f'(x) = 1 - 2x$. Thus $f'(x) = 0$ if and only if $x = \frac{1}{2}$. Since $f''(x) = -2$ is strictly negative everywhere, f is concave down everywhere. It follows that any critical point of f must be a global ~~minimum~~maximum.

For the second part, use the first part (and π in $[0, 1]$) to get

$$0 \leq \pi(1-\pi) = f(\pi) \leq \left(\tfrac{1}{2}\right) = \tfrac{1}{4}.$$

Therefore

$$\sqrt{\frac{\pi(1-\pi)}{n}} \leq \sqrt{\frac{\left(\frac{1}{4}\right)}{n}} = \frac{1}{2\sqrt{n}}.$$

This completes the solution. □

A full solution must give an argument to show that f has a global ~~minimum~~ maximum on $[0, 1]$ at $\frac{1}{2}$. Just proving that there is a local ~~minimum~~ maximum on $[0, 1]$ at $\frac{1}{2}$ is not good enough.

There are other ways to verify this. For example, any quadratic function in which the coefficient of x^2 is strictly negative has a unique critical point, which is a global maximum. Or one can check that $f'(x) > 0$ for $x < \frac{1}{2}$ and $f'(x) < 0$ for $x > \frac{1}{2}$.

Problem 5 (Problem 1.14 in the book)**.** Suppose I toss a fair coin 100 times. (Fair means that the proportion of heads in the population of all tosses is $\pi = \frac{1}{2}$.) Let p be the sample proportion of heads. What is the standard error? What happens to the standard error if instead we use 400 tosses?

Solution. For 100 tosses, we have

$$\mathrm{SE} = \sqrt{\frac{\pi(1-\pi)}{100}} = \sqrt{\frac{\frac{1}{2}(1-\frac{1}{2})}{100}} = \frac{1}{2\sqrt{100}} = \frac{1}{20}.$$

For 400 tosses, we have

$$\mathrm{SE} = \sqrt{\frac{\pi(1-\pi)}{400}} = \sqrt{\frac{\frac{1}{2}(1-\frac{1}{2})}{400}} = \frac{1}{2\sqrt{400}} = \frac{1}{40},$$

which is half as big. □

Problem 6 (Problem 2.10 in the book)**.** Find the grouped mean of the following discrete data:

Class	Frequency
0–4	1
5–7	4
8	3
9–11	2
12–14	2

Solution. We treat it as one occurence of $\frac{1}{2}(4-0) = 2$, four occurences of $\frac{1}{2}(7-5) = 6$, three occurences of 8, two occurences of $\frac{1}{2}(11-9) = 10$, and two occurences of $\frac{1}{2}(14-12) = 13$. This gives

$$\overline{x} = \frac{2 + (4)(6) + (3)(8) + (2)(10) + (2)(13)}{1 + 4 + 3 + 2 + 2} = \frac{96}{12} = 8.$$

This completes the solution. □

Problem 7 (Problem 2.16 in the book)**.** Find the grouped sample standard deviation and estimated standard error of the following discrete data (from Problem 2.10 in the book):

Class	Frequency
0–4	1
5–7	4
8	3
9–11	2
12–14	2

Solution. We already saw that there are 12 observations and that the grouped sample mean is $\overline{x} = 8$. Therefore the grouped sample standard deviation is

$$\begin{aligned} s &= \sqrt{\frac{(2-8)^2 + (4)(6-8)^2 + (3)(8-8)^2 + (2)(10-8)^2 + (2)(13-8)^2}{12-1}} \\ &= \sqrt{\frac{36 + (4)(4) + (3)(0) + (2)(4) + (2)(25)}{11}} \\ &= \sqrt{\frac{110}{11}} = \sqrt{10} \approx 3.16227766. \end{aligned}$$

The estimated standard error is

$$\mathrm{SE} = \frac{s}{\sqrt{12}} = \sqrt{\frac{5}{6}} \approx 0.912870929.$$

This completes the solution. □