QUESTION FOR MATH 343 FOR THE LECTURE OF MONDAY 5 APRIL

Problem 1. Suppose x_1 and x_2 are real numbers. Show that the number b which minimizes $\sqrt{(x_1 - b)^2 + (x_2 - b)^2}$ is $b = \overline{x} = \frac{1}{2}(x_1 + x_2)$.

Do this by using optimization methods from Math 251 on the function $f(b) = (x_1 - b)^2 + (x_2 - b)^2$.

Solution. We calculate

$$f'(b) = 2(x_1 - b)(-1) + 2(x_2 - b)(-1) = 4b - 2(x_1 + x_2).$$

This is zero exactly when $b = \overline{x}$. Moreover, $f''(b) = 4$ is strictly positive everywhere, so f has a unique global minimum, namely at $b = \overline{x}$. $\square$

Problem 2. Suppose $x_1, x_2, \ldots, x_n$ are real numbers. Show that the number b which minimizes

$$\sqrt{\frac{1}{n-1} \sum_{j=1}^{n} (x_j - b)^2}$$

is $b = \overline{x} = \frac{1}{n} \sum_{j=1}^{n} x_j$.

Do this by using optimization methods from Math 251 on the function $f(b) = \sum_{j=1}^{n} (x_j - b)^2$.

Solution. We calculate

$$f'(b) = \sum_{j=1}^{n} 2(x_j - b)(-1) = 2nb - 2 \sum_{j=1}^{n} x_j.$$

This is zero exactly when $b = \overline{x}$. Moreover, $f''(b) = 2n$ is strictly positive everywhere, so f has a unique global minimum, namely at $b = \overline{x}$. $\square$

This shows that the formula for the sample standard deviation, namely

$$s = \sqrt{\frac{1}{n-1} \sum_{j=1}^{n} (x_j - \overline{x})^2},$$

gives a larger answer if $\overline{x}$ is replaced by anything else.

On the other hand, if root mean square error is replaced by something else, the sample mean might no longer minimize the corresponding expression. For example, suppose $x_1 = x_2 = 0$ and $x_3 = 3$. Then $\overline{x} = 1$, but the expression

$$\sqrt[3]{\frac{|x_1 - b|^3 + |x_2 - b|^3 + |x_3 - b|^3}{2}}$$

is minimized when $b = 3\big(\sqrt{2} - 1\big) \approx 1.24$.

Date: 5 April 2021.