MATH 343 (SPRING 2021, PHILLIPS): SOLUTIONS TO SELECTED PROBLEMS IN WRITTEN HOMEWORK 2

The statements of the problems are not identical to what is in the book.

Problem 1 (Problem 3.3 in the book)**.** A fair coin is tossed four times. Find the probabilities of the following events.

(1) Exactly three of the four tosses are heads.
(2) The last two tosses are heads.
(3) The third toss is heads.
(4) All tosses are the same.

Solution. For $k = 1, 2, 3, 4$, let E_k be the event that the k-th toss is heads. These events are all independent, and

$$\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \Pr(E_4) = \frac{1}{2}.$$

Also

$$\Pr(E_1') = \Pr(E_2') = \Pr(E_3') = \Pr(E_4') = \frac{1}{2}.$$

Solution to (1): This asks for the probability of the union of the events

$$E_1 \cap E_2 \cap E_3 \cap E_4', \qquad E_1 \cap E_2 \cap E_3' \cap E_4,$$
$$E_1 \cap E_2' \cap E_3 \cap E_4, \qquad \text{and} \qquad E_1' \cap E_2 \cap E_3 \cap E_4.$$

Since independence is unchanged if some events are replaced by there complements, the events in each of these four intersections are independent. Therefore each has probability $\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$. Since they are disjoint, the probability of their union is the sum of their probabilities. So the answer is

$$\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} = \frac{4}{2^4} = \frac{1}{4}.$$

Solution to (2): This is $\Pr(E_3 \cap E_4)$. Since E_3 and E_4 are independent,

$$\Pr(E_3 \cap E_4) = \Pr(E_3)\Pr(E_4) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Solution to (3): This is $\Pr(E_3) = \frac{1}{2}$.

Solution to (4): This asks for the probability of the union of the events

$$E_1 \cap E_2 \cap E_3 \cap E_4 \qquad \text{and} \qquad E_1' \cap E_2' \cap E_3' \cap E_4'.$$

Since E_1, E_2, E_3, and E_4 are independent, we have

$$\Pr(E_1 \cap E_2 \cap E_3 \cap E_4) = \Pr(E_1)\Pr(E_2)\Pr(E_3)\Pr(E_4) = \left(\frac{1}{2}\right)^4 = \frac{1}{2^4}.$$

Since E_1', E_2', E_3', and E_4' are also independent, we have

$$\Pr(E_1' \cap E_2' \cap E_3' \cap E_4') = \Pr(E_1')\Pr(E_2')\Pr(E_3')\Pr(E_4') = \left(\frac{1}{2}\right)^4 = \frac{1}{2^4}.$$

Date: 14 April 2021.

Since $E_1 \cap E_2 \cap E_3 \cap E_4$ and $E_1' \cap E_2' \cap E_3' \cap E_4'$ are disjoint, we have

$$\begin{aligned}
&\Pr\big([E_1 \cap E_2 \cap E_3 \cap E_4] \cup [E_1' \cap E_2' \cap E_3' \cap E_4']\big) \\
&\qquad = \Pr(E_1 \cap E_2 \cap E_3 \cap E_4) + \Pr(E_1' \cap E_2' \cap E_3' \cap E_4') \\
&\qquad = \frac{1}{2^4} + \frac{1}{2^4} = \frac{1}{2^3} = \frac{1}{8}.
\end{aligned}$$

□

Problem 2 (Problem 3.4 in the book)**.** A fair die is rolled three times. Find the probabilities of the following events.

(1) All the rolls show an even number of dots.
(2) The last two rolls show an even number of dots.
(3) The third roll shows an even number of dots.
(4) Every roll shows a single dot.
(5) Every roll shows the same number of dots.

Solution. Let E_1 be the event that the first roll shows an even number of dots, let E_2 be the event that the second roll shows an even number of dots, and let E_3 be the event that the third roll shows an even number of dots. Then E_1, E_2, and E_3 are independent, as are E_1', E_2', and E_3', and as are E_1', E_2, and E_3, and similarly for all other such combinations. Also, since all 6 rolls are equally likely, and half of them have an even number of dots, we get

$$\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \Pr(E_1') = \Pr(E_2') = \Pr(E_3') = \frac{1}{2}.$$

Solution to (1): This is

$$\Pr(E_1 \cap E_2 \cap E_3) = \Pr(E_1)\Pr(E_2)\Pr(E_3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Solution to (2): This is

$$\Pr(E_2 \cap E_3) = \Pr(E_2)\Pr(E_3) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Solution to (3): This is

$$\Pr(E_3) = \frac{1}{2}.$$

Solution to (4): For $k = 1, 2, 3$, let F_k be the event that the k-th roll shows a single dot. Then F_1, F_2, and F_3 are independent. Also, since all 6 rolls are equally likely, we get and

$$\Pr(F_1) = \Pr(F_2) = \Pr(F_3) = \frac{1}{6}.$$

The quantity we want is

$$\Pr(F_1 \cap F_2 \cap F_3) = \Pr(F_1)\Pr(F_2)\Pr(F_3) = \left(\frac{1}{6}\right)^3 = \frac{1}{6^3}.$$

(If you multiply this out, you get $\frac{1}{216}$.)

Solution to (5): For $m = 1, 2, 3, 4, 5, 6$ let G_m be the event that all three rolls are m. The reasoning of Part (4) shows that $\Pr(G_m) = \frac{1}{6^3}$ for all 6 choices of m.

These events are all disjoint. The quantity we want is

$$\Pr\left(\bigcup_{m=1}^{6} G_m\right) = \sum_{m=1}^{6} \Pr(G_m) = 6\left(\frac{1}{6^3}\right) = \frac{1}{6^2} = \frac{1}{36}.$$

□

Problem 3 (Problem 3.6 in the book)**.** Suppose $\Pr(E) = 0.40$, $\Pr(F) = 0.55$, and $\Pr(E \cap F) = 0.15$. Sketch a Venn diagram and label the probabilities of the events $E \cup F$, $E \cap F'$, $F \cap E'$, and $(E \cup F)'$.

Solution. We have

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F) = 0.40 + 0.55 - 0.15 = 0.8.$$

Since E is the disjoint union of $E \cap F'$ and $E \cap F$, we have

$$\Pr(E \cap F') = \Pr(E) - \Pr(E \cap F) = 0.40 - 0.15 = 0.25.$$

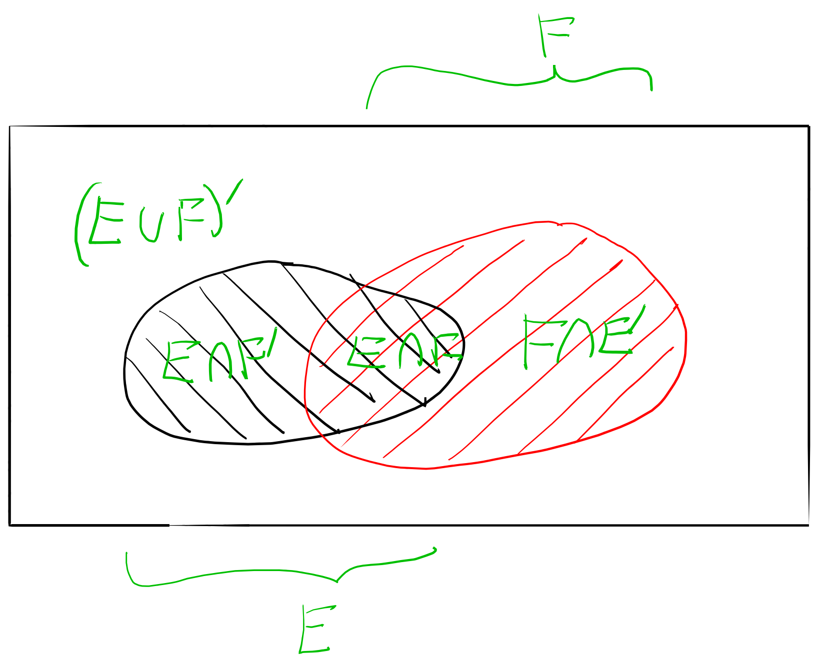
Similarly,

$$\Pr(F \cap E') = \Pr(F) - \Pr(E \cap F) = 0.55 - 0.15 = 0.4.$$

Finally,

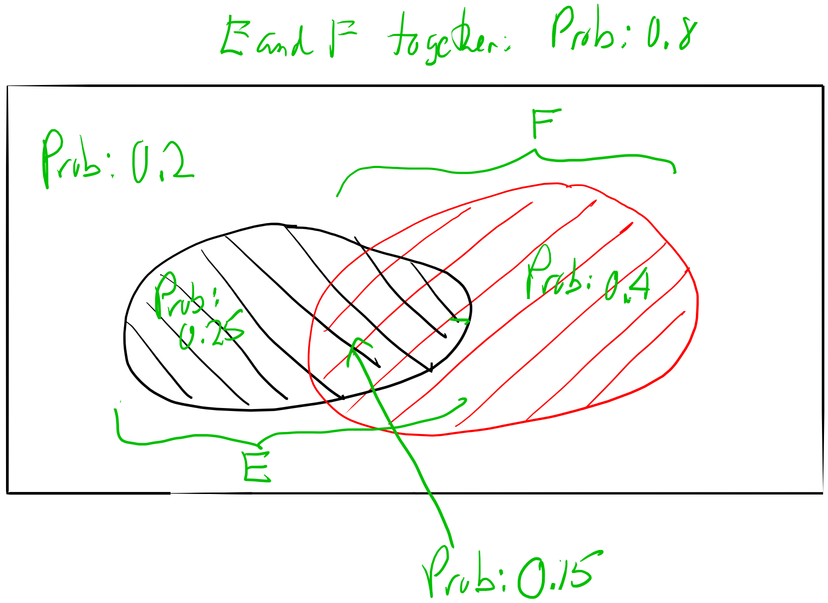
$$\Pr((E \cup F)') = 1 - \Pr(E \cup F) = 1 - 0.8 = 0.2.$$

Here is the diagram:



The black shaded region is E and the red shaded region is F. The labels $E \cap F$, $E \cap F'$, $F \cap E'$, and $(E \cup F)'$ apply only to the smallest region containing them. Thus, $E \cap F'$ is the part that is shaded black but not red. The region $E \cup F$ (not labelled because of awkwardness) is all of the diagram that has at least one kind of shading.

Here is the diagram labelled with probabilities instead:



□

Problem 4 (Problem 3.8 in the book)**.** Suppose $\Pr(E) = 0.55$, $\Pr(F) = 0.40$, and $\Pr(F|E) = 0.20$. Find:

(1) $\Pr(E \cap F)$.
(2) $\Pr(E' \cup F')$.
(3) $\Pr(E' \cap F')$.
(4) $\Pr(E|F)$.

Solution. Solution to (1): We have

$$\Pr(F|E) = \frac{\Pr(F \cap E)}{\Pr(E)},$$

so

$$\Pr(F \cap E) = \Pr(E)\Pr(F|E) = (0.55)(0.20) = 0.11.$$

Solution to (2): We have $E' \cup F' = (F \cap E)'$, so, using Part (1) to get $\Pr(F \cap E)$,

$$\Pr(E' \cup F') = 1 - \Pr(F \cap E) = 1 - 0.11 = 0.89.$$

Solution to (3): Using Part (1) to get $\Pr(F \cap E)$, we have

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F) = 0.55 + 0.40 - 0.11 = 0.84.$$

Now $E' \cap F' = (E \cup F)'$, so

$$\Pr(E' \cap F') = 1 - (E \cup F) = 1 - 0.84 = 0.16.$$

Solution to (4): Using Part (1) to get $\Pr(F \cap E)$, we have

$$\Pr(E|F) = \frac{\Pr(F \cap E)}{\Pr(F)} = \frac{0.11}{0.40} = 0.275.$$

□

Problem 5 (Problem 3.9 in the book)**.** Suppose $\Pr(E) = 0.50$, $\Pr(F) = 0.20$, and $\Pr(F|E) = 0.30$. Find:

(1) $\Pr(E \cap F)$.
(2) $\Pr(E \cup F)$.
(3) $\Pr(E' \cap F')$.
(4) $\Pr(E|F)$.

Solution. Solution to (1): We have

$$\Pr(F|E) = \frac{\Pr(F \cap E)}{\Pr(E)},$$

so

$$\Pr(F \cap E) = \Pr(E)\Pr(F|E) = (0.50)(0.30) = 0.15.$$

Solution to (2): Using Part (1) to get $\Pr(F \cap E)$,

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F) = 0.50 + 0.20 - 0.15 = 0.55.$$

Solution to (3): Using Part (2) to get $\Pr(F \cup E)$, wee have $E' \cap F' = (E \cup F)'$, so

$$\Pr(E' \cap F') = 1 - (E \cup F) = 1 - 0.55 = 0.45.$$

Solution to (4): Using Part (1) to get $\Pr(F \cap E)$, we have

$$\Pr(E|F) = \frac{\Pr(F \cap E)}{\Pr(F)} = \frac{0.15}{0.20} = 0.75.$$

□

Problem 6 (Problem 3.10 in the book)**.** A survey of married couples in a city shows that 20% of the husbands and 8% of the wives watched the 2003 Super Bowl football game. Also, if the husband watched, the probability that the wife watched increased to 25%. Find the probabilities of the following events.

(1) Both the husband and wife watched the game.
(2) At least one of the husband and wife watched the game.
(3) Neither the husband nor the wife watched the game.
(4) The husband watched the game given that the wife watched the game.

Solution. Let H be the event that the husband watched the game, and ket W be the event that the wife watched the game. We are told $\Pr(H) = 0.2$, $\Pr(W) = 0.08$. and $\Pr(W|H) = 0.25$.

Solution to (1): This of $\Pr(H \cap W)$. These events are not independent. Instead:

$$\Pr(H \cap W) = \Pr(H)\Pr(W|H) = (0.2)(0.25) = 0.05.$$

Solution to (2): Using Part (1) to get $\Pr(H \cap W)$, this is

$$\Pr(H \cup W) = \Pr(H) + \Pr(W) - \Pr(H \cap W) = 0.2 + 0.08 - 0.05 = 0.23.$$

Solution to (3): Using Part (1) to get $\Pr(H \cup W)$, this is

$$\Pr(H' \cap W') = \Pr((H \cup W)') = 1 - \Pr(H \cup W) = 0.77.$$

Solution to (4): Using Part (1) to get $\Pr(H \cap W)$, this is

$$\Pr(H|W) = \frac{\Pr(H \cap W)}{\Pr(W)} = \frac{0.05}{0.08} = \frac{5}{8} = 0.625.$$

(I am in a minority. I don't remember what happened in 2003. However, in 2021 my wife watched Super Bowl but I didn't.) □

Problem 7 (Problem 3.12 in the book)**.** A company makes optical lenses under contract to the US military. The lenes are ground to precise specifications and are shipped in lots of 100. Military inspectors check 2 different lenses on each lot of 100. Let E_1 be the event that the first lens fails the inspection and let E_2 be the event that the second lens fails the inspection. If either lens fails, the entire lot of 100 lenses is returned.

Suppose that, in a particular shipment, 3 lenses are bad. Find the probability that the shipment is rejected.

Solution. For the first choice, there are 100 lenses total, of which 97 are good. So the probability of choosing a good one is $\Pr(E_1') = \frac{97}{100}$. Given that the first lens chosen was good, there are 99 lenses left, of which 96 are good. So the probability of choosing a good one is $\Pr(E_2'|E_1') = \frac{96}{99} = \frac{32}{33}$. Therefore the probability that both lenses chosen are good is

$$\Pr(E_1' \cap E_2') = \Pr(E_1') \Pr(E_2'|E_1') = \left(\frac{97}{100}\right)\left(\frac{32}{33}\right) = \frac{97 \cdot 8}{25 \cdot 33}.$$

So the probability that the shipment is rejected is

$$\Pr(E_1 \cup E_2) = 1 - \Pr(E_1' \cap E_2') = 1 - \frac{97 \cdot 8}{25 \cdot 33}.$$

(This is about $1 - 0.9406 = 0.0594$, or about 5.94%.) □

Problem 8 (Problem 3.14 in the book; "Problem of de Méré")**.** Which is more likely: getting at least one roll of 1 in four rolls of a fair die, or getting two 1's at least once in 24 rolls of two fair dice? Find both probabilities.

Solution. First consider one roll of 1 in four rolls of a fair die. For $m = 1, 2, 3, 4$, the event of of rolling a 1 has probability $\frac{1}{6}$, and these events are independent. Therefore their complements all have probability $\frac{5}{6}$ and are also independent. So the probability of never getting a 1 in four rolls of a fair die is $\left(\frac{5}{6}\right)^4$, and the probability of getting at least one roll of 1 is

$$1 - \left(\frac{5}{6}\right)^4 \approx 0.51775.$$

The probability of the event that a single roll of two fair dice gives two 1's is, by independence, the product of the probabilities of the event that the first die comes up 1 and the event that the second die comes up 1, which is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$. Now the probability of never getting two 1's in 24 rolls of two fair dice is, by the same reasoning as above, $\left(\frac{35}{36}\right)^{24}$, so the probability of getting at least one roll of two 1's is

$$1 - \left(\frac{35}{36}\right)^{24} \approx 0.49140.$$

We conclude that getting one 1 in four rolls of a single die is slightly more likely. □