MATH 343 (SPRING 2021, PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 4

The statements of the problems are not identical to what is in the book.
Caution: Not enough proofreading has been done.

Problem 1 (Problem 4.10 in the book)**.** Compute the mean μ and standard deviation σ of the random variable X given by the following table for its mass density function f_X, and with $f_X(x) = 0$ for all other $x \in \mathbb{R}$:

x	$f_X(x)$
0	$\frac{1}{27}$
1	$\frac{6}{27}$
2	$\frac{12}{27}$
3	$\frac{8}{27}$
	$\sum_{x \in \mathbb{R}} f_X(x) = 1$

Solution. We have

$$\mu_X = E(X) = 0 \cdot \left(\frac{1}{27}\right) + 1 \cdot \left(\frac{6}{27}\right) + 2 \cdot \left(\frac{12}{27}\right) + 3 \cdot \left(\frac{8}{27}\right) = 2,$$

$$\begin{aligned} \operatorname{Var}(X) &= E(X^2) - E(X)^2 = E(X^2) - 4 \\ &= 0 \cdot \left(\frac{1}{27}\right) + 1^2 \cdot \left(\frac{6}{27}\right) + 2^2 \cdot \left(\frac{12}{27}\right) + 3^2 \cdot \left(\frac{8}{27}\right) - 4 \\ &= \frac{126}{27} - 4 = \frac{2}{3}, \end{aligned}$$

and

$$\sigma = \sqrt{\operatorname{Var}(X)} = \sqrt{\frac{2}{3}} \approx 0.81650.$$

□

Problem 2 (Problem 4.11 in the book)**.** Compute the mean μ and standard deviation σ of the random variables X, Y, and $X + Y$ described as follows. The space S consists of the 8 three letter sequences in which all the letters are either B or G, the sequences all occur with the same probability, $X(s)$ is the number of times B occurs in s, and $Y(s)$ is the number of times G occurs in s.

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Solution. The random variables X, Y, and $X+Y$ are given by the following table:

s	$X(s)$	$Y(s)$	$X(s)+Y(s)$
BBB	3	0	3
BBG	2	1	3
BGB	2	2	3
BGG	1	2	3
GBB	2	1	3
GBG	1	2	3
GGB	1	2	3
GGG	0	3	3

This in turn gives the following table for the probability mass functions of X and Y, with $f_X(x) = f_Y(x) = 0$ for all other $x \in \mathbb{R}$:

x	$f_X(x)$	$f_Y(x)$
0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{3}{8}$
3	$\frac{1}{8}$	$\frac{1}{8}$

$$\begin{aligned}
\mu_X = E(X) &= 0 \cdot \left(\frac{1}{8}\right) + 1 \cdot \left(\frac{3}{8}\right) + 2 \cdot \left(\frac{3}{8}\right) + 3 \cdot \left(\frac{1}{8}\right) = \frac{3}{2}, \\
\operatorname{Var}(X) &= E(X^2) - E(X)^2 = E(X^2) - \frac{9}{4} \\
&= 0 \cdot \left(\frac{1}{8}\right) + 1^2 \cdot \left(\frac{3}{8}\right) + 2^2 \cdot \left(\frac{3}{8}\right) + 3^2 \cdot \left(\frac{1}{8}\right) - \frac{9}{4} \\
&= \frac{24}{8} - \frac{9}{4} = \frac{3}{4},
\end{aligned}$$

and

$$\sigma_X = \sqrt{\operatorname{Var}(X)} = \frac{\sqrt{3}}{2} \approx 0.86603.$$

Since $f_Y = f_X$, we have

$$\mu_Y = \mu_X = \frac{3}{2} \qquad \text{and} \qquad \sigma_Y = \sigma_X = \frac{\sqrt{3}}{2}.$$

On the other have, $f_Z(3) = 1$ and $f_Z(x) = 0$ for all other $x \in \mathbb{R}$, so $\mu_Z = 3$, $Z - \mu_Z = 0$, $\operatorname{Var}(Z) = E(Z - \mu_Z) = 0$, and $\sigma_Z = \sqrt{\operatorname{Var}(Z)} = 0$. □

Problem 3 (Problem 4.16 in the book)**.** Subjects of a clinical trial of an experimental drug treatment are randomly and uniformly assigned to seven dosage levels U labelled 1 through 7.

(1) Find the probability mass function. (The instructions said to omit the part of the problem which asked for the probability histogram.)
(2) Find μ and σ.
(3) Find $\Pr(\mu - \sigma \leq U \leq \mu + \sigma)$.

Solution. Solution to (1): Clearly $f_U(x) = \frac{1}{7}$ for $x \in \{1, 2, 3, 4, 5, 6, 7\}$ and $f_U(x) = 0$ for all other $x \in \mathbb{R}$.

Solution to (2): We have

$$\mu = E(U) = 1 \cdot \left(\frac{1}{7}\right) + 2 \cdot \left(\frac{1}{7}\right) + 3 \cdot \left(\frac{1}{7}\right) + 4 \cdot \left(\frac{1}{7}\right) + 5 \cdot \left(\frac{1}{7}\right) + 6 \cdot \left(\frac{1}{7}\right) + 7 \cdot \left(\frac{1}{7}\right) = 4,$$

$$E(U^2) = 1^2 \cdot \left(\frac{1}{7}\right) + 2^2 \cdot \left(\frac{1}{7}\right) + 3^2 \cdot \left(\frac{1}{7}\right) + 4^2 \cdot \left(\frac{1}{7}\right) + 5^2 \cdot \left(\frac{1}{7}\right) + 6^2 \cdot \left(\frac{1}{7}\right) + 7^2 \cdot \left(\frac{1}{7}\right) = 20,$$

$$\mathrm{Var}(U) = E(U^2) - E(U)^2 = 20 - 16 = 4,$$

and

$$\sigma = \sqrt{\mathrm{Var}(X)} = 2.$$

Solution to (3): We have $\mu - \sigma \leq U(s) \leq \mu + \sigma$ if and only if s is one of $2, 3, 4, 5, 6$, so $\Pr(\mu - \sigma \leq U \leq \mu + \sigma) = \frac{5}{7} \approx 0.71429$. □

Problem 4 (Problem 4.23 in the book)**.** A lab technician knows that two of the eight pints of blood in the blood bank contain Type A Rh-positive blood. The technician selects two of the pints at random (without replacement). Let K be the number of pints of Type A Rh-positive blood obtained.

(1) Find the probability mass function of K (given as a function or as a table).
(2) Graph the cumulative distribution function of K.
(3) Find the expected value μ_K.
(4) Find the standard deviation σ_K.

Solution. Solution to (1): There are four ways the two pints can turn out. Taking R to mean Type A Rh-positive blood and Z to mean any other kind, we can designate them by the sequences RR, RZ, ZR, and ZZ. The probabilities of these sequences are as follows:

$$\Pr(\mathrm{RR}) = \left(\frac{2}{8}\right)\left(\frac{1}{7}\right) = \frac{1}{28}, \qquad \Pr(\mathrm{RZ}) = \left(\frac{2}{8}\right)\left(\frac{6}{7}\right) = \frac{6}{28},$$

$$\Pr(\mathrm{ZR}) = \left(\frac{6}{8}\right)\left(\frac{2}{7}\right) = \frac{6}{28}, \qquad \text{and} \qquad \Pr(\mathrm{ZZ}) = \left(\frac{6}{8}\right)\left(\frac{5}{7}\right) = \frac{15}{28}.$$

So

$$f_K(0) = \frac{15}{28}, \qquad f_K(1) = \frac{6}{28} + \frac{6}{28} = \frac{12}{28}, \qquad f_K(2) = \frac{1}{28},$$

and $f_K(x) = 0$ for all other $x \in \mathbb{R}$.

If you prefer this form, the table is

x	$f_K(x)$
0	$\frac{15}{28} \approx 0.53571$
1	$\frac{12}{28} \approx 0.42857$
2	$\frac{1}{28} \approx 0.03571,$

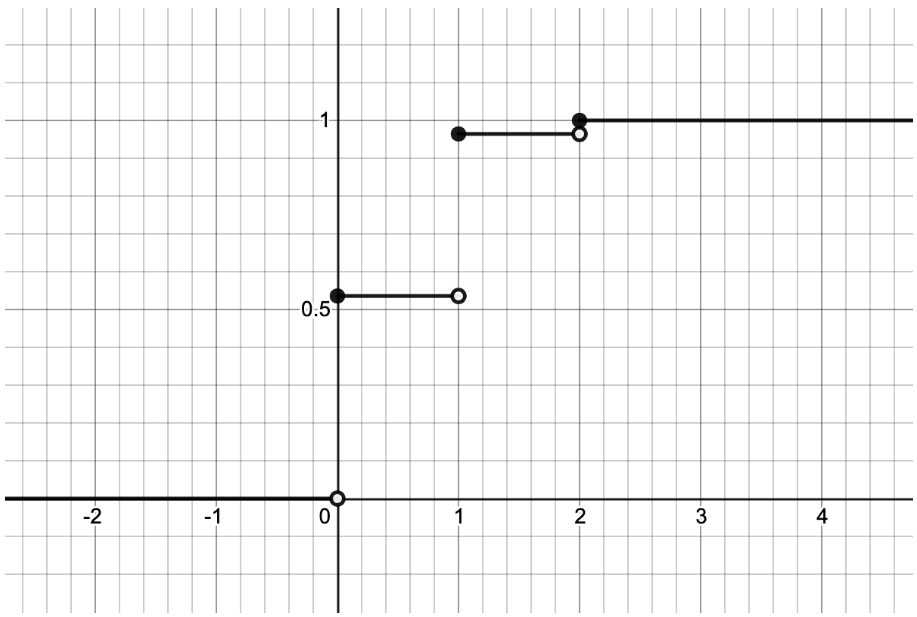
with $f_K(x) = 0$ for all other $x \in \mathbb{R}$.

Solution to (2): The cumulative distribution function is given by

$$F_K(x) = \begin{cases} 0 & x < 0 \\ \frac{15}{28} & 0 \leq x < 1 \\ \frac{27}{28} & 1 \leq x < 2 \\ 1 & 2 < x. \end{cases}$$

The number $\frac{27}{28}$ is obtained as $f_K(0) + f_K(1) = \frac{15}{28} + \frac{12}{28}$.

Here is the graph.



The open circles and filled in circles must be correct for a fully correct solution.

Solution to (3): We have

$$\mu_K = E(K) = 0 \cdot \left(\frac{15}{28}\right) + 1 \cdot \left(\frac{12}{28}\right) + 2 \cdot \left(\frac{1}{28}\right) = \frac{1}{2}.$$

Solution to (4): We have

$$E(K^2) = 0 \cdot \left(\frac{15}{28}\right) + 1^2 \cdot \left(\frac{12}{28}\right) + 2^2 \cdot \left(\frac{1}{28}\right) = \frac{16}{28}.$$

Therefore

$$\mathrm{Var}(K) = E(K^2) - E(K)^2 = \frac{9}{28}$$

and

$$\sigma_K = \sqrt{\mathrm{Var}(K)} = \sqrt{\frac{9}{28}} = \frac{3}{2\sqrt{7}} \approx 0.56695.$$

□

Problem 5 (Problem 4.26 in the book)**.** Consider families of four children. Assume that boy and girls are equally likely and independent. Let X be the absolute value of the difference between the number of boys and the number of girls.

(1) Find the probability mass function of X (given as a table).
(2) Find the cumulative distribution function of X and sketch its graph.
(3) Find the mean of X.
(4) Find the standard deviation of X.

Solution. Solution to (1): There are 16 equally likely sequences. Letting B represent boy and G represent girl, they are BBBB, BBBG, etc.

One can make a list of all 16 of them, and calculate the absolute value of the difference for each one, but it is simpler to do the following. First, there are exactly two sequences s for which $X(s) = 4$, namely $s =$ BBBB and $s =$ GGGG. There are four sequences in which B occurs once, namely BGGG, GBGG, GGBG, and GGGB, and similarly four sequences in which G occurs once. For each of these, $X(s) = 2$. In the remaining sequences, B and G each occur twice, giving $X(s) = 0$. Therefore

$$f_X(0) = \frac{6}{16} = \frac{3}{8}, \qquad f_X(2) = \frac{8}{16} = \frac{4}{8} = \frac{1}{2}, \qquad f_X(4) = \frac{2}{16} = \frac{1}{8},$$

and $f_X(x) = 0$ for all other $x \in \mathbb{R}$. The table is thus

x	$f_K(x)$
0	$\frac{3}{8}$
2	$\frac{1}{2}$
4	$\frac{1}{8}$,

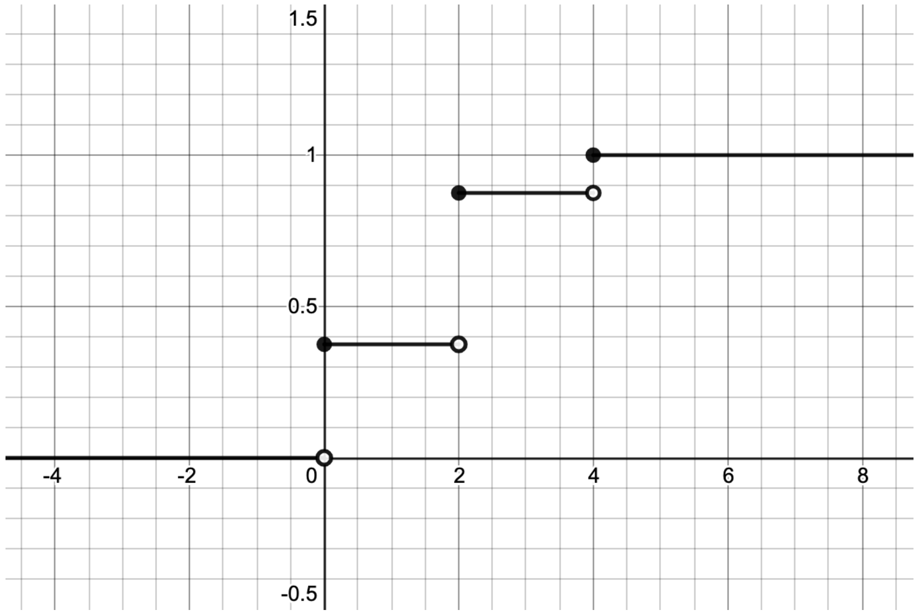
with $f_X(x) = 0$ for all other $x \in \mathbb{R}$.

Solution to (2): The cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{8} & 0 \le x < 2 \\ \frac{7}{8} & 2 \le x < 4 \\ 1 & 4 < x. \end{cases}$$

The number $\frac{7}{8}$ is obtained as $f_X(0) + f_X(2) = \frac{3}{8} + \frac{4}{8}$.

Here is the graph.



The open circles and filled in circles must be correct for a fully correct solution.

Solution to (3): We have

$$\mu_X = E(X) = 0 \cdot \left(\frac{3}{8}\right) + 2 \cdot \left(\frac{4}{8}\right) + 4 \cdot \left(\frac{1}{8}\right) = \frac{3}{2} = 1.5.$$

Solution to (4): We have

$$E(X^2) = 0 \cdot \left(\frac{3}{8}\right) + 2^2 \cdot \left(\frac{4}{8}\right) + 4^2 \cdot \left(\frac{1}{8}\right) = 4.$$

Therefore

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = 4 - \frac{9}{4} = \frac{7}{4}$$

and

$$\sigma_X = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2} \approx 1.32288.$$

□

Problem 6 (Problem 4.28 in the book)**.** Let U be the number of dots on the roll of a fair die. Find the following quantities.

(1) $E(U)$.
(2) $E(2U - 7)$.
(3) $E(U^2 + 4)$.
(4) $\mathrm{Var}(U)$.

Solution. Solution to (1):

We have $f_U(x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$, and $f_X(x) = 0$ for all other $x \in \mathbb{R}$. Therefore

$$E(U) = 1 \cdot \left(\frac{1}{6}\right) + 2 \cdot \left(\frac{1}{6}\right) + 3 \cdot \left(\frac{1}{6}\right) + 4 \cdot \left(\frac{1}{6}\right) + 5 \cdot \left(\frac{1}{6}\right) + 6 \cdot \left(\frac{1}{6}\right) = \frac{7}{2}.$$

Solution to (2): We have $E(2U - 7) = 2E(U) - 7 = 0$.

Solution to (3): We have

$$E(U^2) = 1^2 \cdot \left(\frac{1}{6}\right) + 2^2 \cdot \left(\frac{1}{6}\right) + 3^2 \cdot \left(\frac{1}{6}\right) + 4^2 \cdot \left(\frac{1}{6}\right) + 5^2 \cdot \left(\frac{1}{6}\right) + 6^2 \cdot \left(\frac{1}{6}\right) = \frac{91}{6}.$$

Therefore

$$E(U^2 + 4) = \frac{91}{6} + 4 = \frac{115}{6} \approx 19.16667.$$

Solution to (4): We have

$$\mathrm{Var}(U) = E(U^2) - E(U)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.91667.$$

□

Problem 7 (Problem 4.30 in the book)**.** Two fair dice are rolled. Let X be the smaller of the two numbers rolled.

(1) Find the probability mass function of X (given as a table).
(2) Find the mean of X.
(3) Find the standard deviation of X.

Solution. Solution to (1): Suppose one die is red and the other is green. The smaller number is 1 if and only if the red die comes up 1 or the green die comes up 1. There are 6 ways for the first to happen and 6 ways for the second to happen, but the combination $(1, 1)$ has been counted twice, so there are $6 + 6 - 1 = 11$ ways for the smaller number to be 1.

The smaller number is 2 if and only if the red die comes up 2 and the green die comes up as one of $2, 3, 4, 5, 6$, or the green die comes up 2 and the red die comes up as one of $2, 3, 4, 5, 6$. There are 5 ways for the first to happen and 5 ways for the second to happen, but the combination $(2, 2)$ has been counted twice, so there are $5 + 5 - 1 = 9$ ways for the smaller number to be 2.

Similarly, there are $4 + 4 - 1 = 7$ ways for the smaller number to be 3, there are $3 + 3 - 1 = 5$ ways for the smaller number to be 4, there are $2 + 2 - 1 = 3$ ways for the smaller number to be 5, and there is only 1 way for the smaller number to be 6, namely $(6, 6)$.

All 36 outcomes are equally probable, so we get the table

x	$f_X(x)$
1	$\frac{11}{36}$
2	$\frac{9}{36}$
3	$\frac{7}{36}$
4	$\frac{5}{36}$
5	$\frac{3}{36}$
6	$\frac{1}{36}$,

with $f_X(x) = 0$ for all other $x \in \mathbb{R}$.

Solution to (2): We have

$$\begin{aligned}\mu &= E(X)\\ &= 1\cdot\left(\frac{11}{36}\right)+2\cdot\left(\frac{9}{36}\right)+3\cdot\left(\frac{7}{36}\right)+4\cdot\left(\frac{5}{36}\right)+5\cdot\left(\frac{3}{36}\right)+6\cdot\left(\frac{1}{36}\right)\\ &= \frac{91}{36}\approx 2.52778.\end{aligned}$$

Solution to (3): We have

$$\begin{aligned}E(X^2) &= 1^2\cdot\left(\frac{11}{36}\right)+2^2\cdot\left(\frac{9}{36}\right)+3^2\cdot\left(\frac{7}{36}\right)+4^2\cdot\left(\frac{5}{36}\right)+5^2\cdot\left(\frac{3}{36}\right)+6^2\cdot\left(\frac{1}{36}\right)\\ &= \frac{301}{36}\approx 8.36111,\end{aligned}$$

so

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2555}{36^2} \approx 1.97145$$

and

$$\sigma_X = \sqrt{\mathrm{Var}(X)} = \frac{\sqrt{2555}}{36} \approx 1.40409.$$

□

Problem 8 (Problem 5.2 in the book; misprint corrected)**.** Consider Mendel's experiment as described in Example 5.4. Encode a red flowered plant with 1 and a white flowered plant with 0. Suppose that when a second generation plant is selected from the crossing of a pea plant with red flowers and one with white flowers, the probability of obtaining a pea plant with red flowers is $\frac{3}{4}$. Find μ and σ.

Solution. Using the formulas in the book,

$$\mu = \frac{3}{4} \qquad \text{and} \qquad \sigma = \sqrt{\left(\frac{3}{4}\right)\left(1-\frac{3}{4}\right)} = \frac{\sqrt{3}}{4} \approx 0.44301.$$

□

Problem 9 (Problem 5.4 in the book)**.** Let X be a Bernoulli random variable with parameter π. Find $E((X-\mu_X)^k)$ for $k = 1, 2, 3$.

Solution. Recall that

$$f_X(x) = \begin{cases} 1-\pi & x = 0\\ \pi & x = 1\\ 0 & \text{otherwise.}\end{cases}$$

We know that $\mu = \pi$. Let $Y_k = (X-\mu_X)^k$. Then

$$f_{Y_1}(x) = \begin{cases} 1-\pi & x = -\pi\\ \pi & x = 1-\pi\\ 0 & \text{otherwise,}\end{cases}$$

$$f_{Y_2}(x) = \begin{cases} 1-\pi & x = (-\pi)^2\\ \pi & x = (1-\pi)^2\\ 0 & \text{otherwise,}\end{cases}$$

and

$$f_{Y_3}(x) = \begin{cases} 1-\pi & x = (-\pi)^3 \\ \pi & x = (1-\pi)^3 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$E(Y_1) = (1-\pi)(-\pi) + \pi(1-\pi) = 0,$$
$$E(Y_2) = (1-\pi)(-\pi)^2 + \pi(1-\pi)^2 = \pi - \pi^2 = \pi(1-\pi),$$

and

$$E(Y_2) = (1-\pi)(-\pi)^3 + \pi(1-\pi)^3 = \pi - 3\pi^2 + 2\pi^3.$$

□

Alternate solution. We have

$$E(X-\mu) = E(X) - \mu = \mu - \mu = 0.$$

(This is true for any random variable with finite mean.)

Next, using $X^2 = X$ (since X only takes the values 0 and 1, and $\mu_X = \pi$ (from the book), we get

$$\begin{aligned} E((X-\mu_X)^2) &= E(X^2 - 2\mu X + \mu^2) = E(X - 2\mu X + \mu^2) = E((1-2\mu)X + \mu^2) \\ &= (1-2\mu)E(X) + \mu^2 = (1-2\mu)\mu + \mu^2 = \mu - \mu^2 = \pi - \pi^2. \end{aligned}$$

Similarly,

$$\begin{aligned} E((X-\mu_X)^3) &= E(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3) = E(X - 3\mu X + 3\mu^2 X - \mu^3) \\ &= E([1 - 3\mu + 3\mu^2]X - \mu^3) = [1 - 3\mu + 3\mu^2]E(X) - \mu^3 \\ &= [1 - 3\pi + 3\pi^2]\pi - \pi^3 = \pi - 3\pi^2 + 2\pi^3. \end{aligned}$$

□