Continuous Random Variables

- Ex. Height of a randomly chosen UO student. (As modeled, there are actually only finitely many students.)
- Ex. Location of an accident on I-5 between Eugene and Roseburg, measured in miles from Eugene.
- Ex. If a cumulative distribution for RV with discrete + cont. parts.

\[
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F_X(x) &= \begin{cases} 
0 & x < 0 \\
\frac{x}{5} & 0 \leq x < 5 \\
1 & x \geq 5
\end{cases}
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Ex. Unit distr. on \([0, 3]\) \( \mu = \frac{1}{3} \) \( \sigma = 1 \).

\[
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\frac{x}{3} & 0 \leq x < 3 \\
1 & x \geq 3
\end{cases}
\]

Median: Should be 4.

\[
F_X(4) = \frac{1}{2}
\]
Ex. \( f_X(x) = \int_0^x e^{-t} \, dt \) \( 0 \leq x \leq 1 \) \( \text{Find } M. \)

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\( e^{-M} = \frac{1}{2} \) \( \text{so } M = -\ln \left( \frac{1}{2} \right) = (\ln 2). \)

Expected value \( E(X) = \int_0^\infty x f_X(x) \, dx \)

Also \( M_X \) \( \text{Recall in discrete we } E(X) = \sum x f_X(x) \)

Ex. \( \lim_{t \to 0} \frac{f_X(t)}{t} = f_X(0) = \int_0^x \frac{1}{6} \, dx \)

\( E(X) = \int_0^\infty x \cdot \frac{1}{6} \, dx = \frac{1}{6} \cdot \frac{x^2}{2} \bigg|_0^6 = \frac{1}{6} \cdot 6^2 - \frac{1}{6} \cdot 0 = 4. \)

Ex. \( f_X \) as at top of page:

\( E(X) = \int_0^\infty xe^{-x} \, dx = x(-e^{-x}) \bigg|_0^\infty = \int_0^\infty 1 \cdot (-e^{-x}) \, dx \)

Integrate by parts \( u(x) = x, \quad v'(x) = e^{-x} \)

\( u'(x) = 1, \quad v(u) = -e^{-x} \quad \text{by substitution } w = -x \)

\( -xe^{-x} \bigg|_0^\infty + \int_0^\infty e^{-x} \, dx = 1. \)

With substitution \( w = -x, \) find that \( du = -dx. \)

\( \int e^{-x} \, dx \quad w = -x, \quad dx = -dw \)

\( = \int e^w \, dw = -e^w + C = -e^{-x} + C. \)

\( \int_0^\infty e^{-x} \, dx = (-e^{-x}) \bigg|_0^\infty = 0 - (-1) = 1. \)

\( f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \)

\( E(X) = \int_0^\infty x f_X(x) \, dx = \int_0^1 t \cdot 2t \, dt = \cdots = \frac{2}{3}. \)

Median is: \( F_X(x) = \int_{-\infty}^x f_X(t) \, dt = \int_0^x 2t \, dt = x^2. \) This is \( \frac{1}{2} \) when \( x = \frac{1}{2}. \)

\( \Pr (X \leq x) \) \( \text{if } 0 \leq x \leq 1 \)

\( \text{Then } E(g(X)) = \int_{-\infty}^\infty g(x) f_X(x) \, dx. \)

Ex. \( X \) is the distance of an accident from Eugene in miles.

\( 5280X \) is the distance in feet.
Just as for discrete case. If \( a, b \) are constants, then \( E(aX + b) = aE(X) + b \)
\[ \mu_X = E(X) \quad \text{Var}(X) = E((X - \mu_X)^2) \]
Thus \( \text{Var}(X) = E(X^2) - E(X)^2 \)

**Std dev**: \( \sigma_X = \sqrt{\text{Var}(X)} \).

**Normal distribution**: \( Z \sim \text{N}(\mu, \sigma) \quad \mu \in (-\infty, \infty), \sigma > 0 \)

\[ f_Z(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2}, \quad f_Z\mu\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \]

\[ E(Z) = \int_{-\infty}^{\infty} x (\frac{1}{\sqrt{2\pi}\sigma}) e^{-x^2/2} dx \]

This is an odd function, so \( \int_{-a}^{a} x (\frac{1}{\sqrt{2\pi}\sigma}) e^{-x^2/2} dx = 0 \)

One ought to check that \( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 \)

This comes from \( \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \) (1)

**Warning**: \( e^{-x^2} \) has an antiderivative, but it is not an elementary function.

\[ E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \]

Parts: \( u(x) = -x, \quad v(x) = -xe^{-x^2/2} \)

\[ u'(x) = -1, \quad v(x) = -\frac{1}{2} e^{-x^2/2} \]

More steps: \( \int (-xe^{-x^2/2}) dx \)

\[ w = \frac{x^2}{2}, \quad dw = -xdx \]

\[ = \int e^w dw = e^w + C = e^{-x^2/2} + C \]

For \( \int_{-\infty}^{\infty} e^{-x^2} dx \). Consider a solid of revolution:

Between graph of \( e^{-x^2} \) and \( x \)-axis, rotated about \( y \)-axis.

Volume using shells:

\[ 2\pi \int_0^\infty xe^{-x^2} dx \]

\[ \text{Height} \times \text{thickness} \]

\[ \text{Circumference} \]

\[ \text{Volume} \int_0^\infty 2\pi xe^{-x^2} dx = 15(-e^{-x})\bigg|_0^\infty = \pi \quad \text{substitute} \ u = -x^2 \]
Top view: At \((x, y)\) get \(e^{-\ell^2x^2} \sin \ell r = e^{-\ell^2x^2} \sin \ell \sqrt{r^2 - x^2}\). If \(\text{let it a point at distance } r \text{ from } (0, 0)\), height is \(e^{-r^2}\).

On circle, get height \(e^{-r^2}\) (just because we rotated about vertical axis at \((0, 0)\)).

Volume over thickened red line is
\[
\left(\int_{-\infty}^{\infty} e^{-c^2-x^2} \, dx\right) \cdot dc = e^{-c^2} \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx\right) \cdot dc
\]

Now integrate with respect to \(c\):
\[
\int_{-\infty}^{\infty} e^{-c^2} \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx\right) \, dc = \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx\right)^2
\]

This is a constant
\[
\Rightarrow \text{Area we are sure.}
\]

Putting these together: \(\left(\int_{-\infty}^{\infty} e^{-x^2} \, dx\right)^2 = \pi\).