MATH 343 (SPRING 2021, PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 6

The statements of the problems are not identical to what is in the book.

Problem 1 (Problem 5.8 in the book). In a binomial experiment with each of the following pairs of values of \( n \) and \( \pi \), let \( K \) be the random variable which counts the number of successes in \( n \) trials with probability \( \pi \) of success, and find \( \mu_K \) and \( \sigma_K \).

1. \( n = 100 \) and \( \pi = 0.40 \).
2. \( n = 400 \) and \( \pi = 0.40 \).
3. \( n = 100 \) and \( \pi = 0.60 \).
4. \( n = 400 \) and \( \pi = 0.60 \).

Solution. For all parts we use the formulas from the textbook

\[
\mu_K = n\pi \quad \text{and} \quad \sigma_K = \sqrt{n\pi(1-\pi)}.
\]

The approximations are provided so that people who used calculators can check their approximate answers.

Solution to (1):

\[
\mu_K = n\pi = 100(0.40) = 40
\]

and

\[
\sigma_K = \sqrt{n\pi(1-\pi)} = \sqrt{100(0.4)(0.6)} = \sqrt{24} = 2\sqrt{6} \approx 4.8990.
\]

Solution to (2):

\[
\mu_K = n\pi = 400(0.40) = 160
\]

and

\[
\sigma_K = \sqrt{n\pi(1-\pi)} = \sqrt{400(0.4)(0.6)} = \sqrt{96} = 4\sqrt{6} \approx 9.79780.
\]

Solution to (3):

\[
\mu_K = n\pi = 100(0.60) = 60
\]

and

\[
\sigma_K = \sqrt{n\pi(1-\pi)} = \sqrt{100(0.6)(0.4)} = \sqrt{24} = 2\sqrt{6} \approx 4.8990.
\]

Solution to (4):

\[
\mu_K = n\pi = 400(0.60) = 240
\]

and

\[
\sigma_K = \sqrt{n\pi(1-\pi)} = \sqrt{400(0.6)(0.4)} = \sqrt{96} = 4\sqrt{6} \approx 9.79780.
\]

\hfill \Box

Problem 2 (Problem 5.10 in the book). In a test of extrasensory perception (ESP), a subject must guess which one of three cards lying face down on a table is the ace of spades. Suppose that the subject has no ESP and is just guessing. Find the probability that in 12 guesses, the number of cards guessed correctly is each of the following.

1. Exactly 4.
2. Exactly 5.

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(3) Exactly 6.
(4) Less than 6.
(5) At least 6.
(6) More than 6.

Solution. In all parts, we use the binomial distribution for the number of successes in 12 trials with probability \( \frac{1}{3} \) of success. The approximations are provided so that people who used calculators can check their approximate answers.

Solution to (1):
\[
\binom{12}{4} \left( \frac{1}{3} \right)^4 \left( 1 - \frac{1}{3} \right)^8 = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \left( \frac{2^8}{3^{12}} \right) = \frac{495 \cdot 2^8}{3^{12}} \approx 0.23845.
\]
Solution to (2):
\[
\binom{12}{5} \left( \frac{1}{3} \right)^5 \left( 1 - \frac{1}{3} \right)^7 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \left( \frac{2^7}{3^{12}} \right) = \frac{792 \cdot 2^7}{3^{12}} \approx 0.19076.
\]
Solution to (3):
\[
\binom{12}{6} \left( \frac{1}{3} \right)^6 \left( 1 - \frac{1}{3} \right)^6 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \left( \frac{2^6}{3^{12}} \right) = \frac{924 \cdot 2^6}{3^{12}} \approx 0.11127.
\]
Solution to (4): Add up the probabilities of the values 0, 1, 2, 3, 4, and 5, getting
\[
\sum_{k=0}^{5} \binom{12}{k} \left( \frac{1}{3} \right)^k \left( 1 - \frac{1}{3} \right)^{12-k} = \sum_{k=0}^{5} \binom{12}{k} \left( \frac{2^{n-k}}{3^{12}} \right) \approx 0.82228.
\]
Solution to (5): The event here is the complement of the event in (5). Therefore the answer is
\[
1 - \sum_{k=0}^{5} \binom{12}{k} \left( \frac{1}{3} \right)^k \left( 1 - \frac{1}{3} \right)^{12-k} \approx 1 - 0.82228 = 0.17772.
\]
(One can also add up the probabilities of the values 6, 7, 8, 9, 9, 11, and 12, getting
\[
\sum_{k=6}^{12} \binom{12}{k} \left( \frac{1}{3} \right)^k \left( 1 - \frac{1}{3} \right)^{12-k} = \sum_{k=6}^{12} \binom{12}{k} \left( \frac{2^{n-k}}{3^{12}} \right) \approx 0.93355,
\]
but this takes much longer.)
Solution to (6): Subtract the result of (3) from the result of (5), getting approximately 0.17772 – 0.11127 = 0.06644.
(One can also add up the probabilities of the values 7, 8, 9, 10, 11, and 12, getting
\[
\sum_{k=7}^{12} \binom{12}{k} \left( \frac{1}{3} \right)^k \left( 1 - \frac{1}{3} \right)^{12-k} = \sum_{k=7}^{12} \binom{12}{k} \left( \frac{2^{n-k}}{3^{12}} \right) \approx 0.066448,
\]
but this takes longer.)

\[\square\]

**Problem 3** (Problem 5.11 in the book). Only about 10% of people survive an infection of the Ebola virus. Assume this is exact. Among 10 independent cases of Ebola, what is the probability that the number of survivors is each of the following.

(1) Exactly 0.
(2) Exactly 1.
(3) Exactly 2.
(4) Less than 2.
(5) At least 2.
(6) More than 2.

Solution. In all parts, we use the binomial distribution for the number of successes in 10 trials with probability \( \frac{1}{10} \) of success. The approximations are provided so that people who used calculators can check their approximate answers.

Solution to (1): This is immediate, namely

\[
\left(1 - \frac{1}{10}\right)^{10} = \frac{9^{10}}{10^{10}} \approx 0.34868.
\]

Solution to (2):

\[
\binom{10}{2} \left(\frac{1}{10}\right)^2 \left(1 - \frac{1}{10}\right)^8 = 10 \cdot 9 \cdot \frac{9^8}{10^{10}} = \frac{9^9}{2 \cdot 10^9} \approx 0.19371.
\]

Solution to (3):

\[
\binom{10}{2} \left(\frac{1}{10}\right)^2 \left(1 - \frac{1}{10}\right)^8 = 10 \cdot 9 \cdot \frac{9^8}{10^{10}} = \frac{9^9}{2 \cdot 10^9} \approx 0.19371.
\]

Solution to (5): Using the answer to (4), this is

\[
1 - \frac{19 \cdot 9^9}{10^{10}} \approx 0.26390.
\]

Solution to (6): Using the answers to (3) and (4), this is

\[
1 - \frac{9^9}{2 \cdot 10^9} - \frac{19 \cdot 9^9}{10^{10}} = 1 - \frac{24 \cdot 9^9}{10^{10}} \approx 0.07029.
\]

One can get this result using other combinations of the previous parts as well. □

Problem 4 (Problem 7.1 in the book). Find the probability density function for the continuous random variable \( X \) with cumulative distribution function

\[
F_X(x) = \begin{cases} 
0 & x < 0 \\
\frac{x^2}{4} & 0 \leq x < 2 \\
1 & 2 \leq x.
\end{cases}
\]

Solution. Differentiate the function \( F_X \). We can assign arbitrary values at the points \( x = 0 \) and \( x = 2 \) (although in fact \( F'_X(0) = 0 \)). Using

\[
\frac{d}{dx} \left(\frac{x^2}{4}\right) = \frac{x}{2},
\]

We get

\[
f_X(x) = F'_X(x) = \begin{cases} 
0 & x < 0 \\
\frac{x}{2} & 0 \leq x < 2 \\
0 & 2 \leq x.
\end{cases}
\]

The most obvious alternative is to take \( f_X(2) = 4 \) instead of \( f_X(2) = 0 \).
Problem 5 (Problem 7.4 in the book). Let $X$ be the continuous random variable with probability density function
\[ f_X(x) = \begin{cases} 
|x| & -1 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases} \]

Find each of the following.

1. The cumulative distribution function $F_X$.
2. $F_X(-0.50)$.
3. $\Pr(X \geq 0)$.
4. $F_X(0.75)$.
5. $\Pr(-0.50 < X \leq 0.75)$.

Solution. Solution to (1): When $x \leq -1$ we have
\[
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt = \int_{-\infty}^{-1} 0 \, dt = 0.
\]

When $-1 < x \leq 0$ we have
\[
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt = \int_{-\infty}^{-1} f_X(t) \, dt + \int_{-1}^{x} f_X(t) \, dt
= F_X(-1) + \int_{-1}^{x} (-t) \, dt
= 0 + \left[ -\frac{t^2}{2} \right]_{-1}^{x} = \frac{1 - x^2}{2}.
\]

(Parentheses are required in the expression $\int_{-1}^{x} (-t) \, dt$. For example, the expression $\int_{-1}^{x} -t \, dt$ is wrong.)

When $0 < x \leq 1$ we have
\[
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt = \int_{-\infty}^{0} f_X(t) \, dt + \int_{0}^{x} f_X(t) \, dt
= F_X(0) + \int_{0}^{x} t \, dt = \frac{1}{2} + \frac{x^2}{2} \bigg|_{0}^{x}
= \frac{1}{2} + \frac{x^2}{2} - 0 = \frac{1 + x^2}{2}.
\]

When $1 < x$ we have
\[
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt = \int_{-\infty}^{1} f_X(t) \, dt + \int_{1}^{x} f_X(t) \, dt
= F_X(1) + \int_{1}^{x} 0 \, dt = 1 + 0 = 1.
\]
Thus

\[
F_X(x) = \begin{cases} 
0 & x \leq -1 \\
1 - x^2 & -1 < x \leq 0 \\
\frac{2}{1 + x^2} & 0 < x \leq 1 \\
1 & 1 < x. 
\end{cases}
\]

It is not permissible to use \( x \) both as a limit of integration and as the variable of integration. For example, the expression

\[
\int_{-\infty}^{x} f_X(x) \, dx
\]

is wrong.

Solution to (2): Using (1), we have

\[
F_X(-0.50) = \frac{1 - (-0.50)^2}{2} = \frac{3}{8} = 0.375.
\]

Solution to (3): Using (1) at the second step, we have

\[
Pr(X \geq 0) = 1 - F_X(0) = \frac{1 - 0^2}{2} = \frac{1}{2}.
\]

(This is also obvious by symmetry.)

Solution to (4): Using (1), we have

\[
F_X(0.75) = \frac{1 + (0.75)^2}{2} = \frac{25}{32} = 0.78125.
\]

Solution to (5): Using (1) at the second step, we have

\[
Pr(-0.50 < X \leq 0.75) = F_X(0.75) - F_X(-0.50) = \frac{1 + (0.75)^2}{2} - \frac{1 - (-0.50)^2}{2} = \frac{13}{32} = 0.40625.
\]

One can also calculate directly:

\[
\int_{-0.5}^{0.75} f_X(t) \, dt = \int_{-0.5}^{0} f_X(t) \, dt + \int_{0}^{0.75} f_X(t) \, dt = \int_{-0.5}^{0} (-t) \, dt + \int_{0}^{0.75} t \, dt,
\]

but this duplicates earlier work. \(\Box\)

**Problem 6** (Problem 7.8 in the book). While driving on a country road, the time \( T \) between oncoming cars is a continuous random variable for which there is a real number \( \theta > 0 \) such that \( f_T(t) = \theta^{-1} e^{-t/\theta} \) when \( t \geq 0 \) and \( f_T(t) = 0 \) when \( t < 0 \). Suppose that the average time between oncoming cars is \( \theta = 2 \) minutes. Find each of the following.

1. \( F_T(2) \).
2. \( Pr(T > 2) \).
3. \( Pr(2 < T \leq 6) \).

**Solution.** It is easiest to first find \( F_T(x) \) for all \( x \). When \( x \leq 0 \) we have

\[
F_T(x) = \int_{-\infty}^{x} f_T(t) \, dt = \int_{-\infty}^{x} 0 \, dt = 0.
\]
When \(0 < x\) we have, using the substitution \(u = -\theta^{-1}t\) at the fourth step,

\[
F_T(x) = \int_{-\infty}^{x} f_T(t) \, dt = \int_{-\infty}^{0} f_T(t) \, dt + \int_{0}^{x} f_T(t) \, dt
\]

\[
= F_T(0) + \int_{0}^{x} \theta^{-1}e^{-t/\theta} \, dt
\]

\[
= 0 + \left( -e^{-t/\theta} \right) \bigg|_{0}^{x} = -e^{-x/\theta} + e^{0/\theta} = 1 - e^{-x/\theta}.
\]

Putting \(\theta = 2\) gives \(F_T(x) = 1 - e^{-x/2}\).

It is not permissible to use \(x\) both as a limit of integration and as the variable of integration. For example, the expression

\[
\int_{-\infty}^{x} f_X(x) \, dx
\]

is wrong.

Solution to (1): We have \(F_T(2) = 1 - \frac{1}{e}\). (Done directly, it is \(\int_{0}^{2} \frac{1}{2}e^{-t/2} \, dt\). This is approximately 0.63212.

Solution to (2): We have \(\Pr(T > 2) = 1 = \Pr(T \leq 2) = 1 - F_T(2) = \frac{1}{e}\). This is approximately 0.36788.

Solution to (3): We have

\[
\Pr(2 < T \leq 6) = F_T(6) - F_T(2) = 1 - \frac{1}{e^3} - \left(1 - \frac{1}{e}\right) = \frac{1}{e} - \frac{1}{e^3} \approx 0.31809.
\]

(Done directly, it is \(\int_{2}^{6} \frac{1}{2}e^{-t/2} \, dt\).)

\[\square\]

Problem 7 (Problem 7.9 in the book). Find the 25th percentile and median for the continuous random variable \(X\) with cumulative distribution function

\[
F_X(x) = \begin{cases} 
0 & x < 0 \\
\frac{x^2}{4} & 0 \leq x < 2 \\
1 & 2 \leq x.
\end{cases}
\]

Solution. The 25th percentile \(p\) is found by solving the equation \(F_X(p) = \frac{1}{4}\). Clearly \(p\) must be in \((0, 2)\), so this equation is

\[
\frac{p^2}{4} = \frac{1}{4}.
\]

There are two solutions, namely \(p = \pm 1\). We reject \(-1\) since it isn’t in \((0, 2)\). (I must see you do this, since otherwise I don’t know you solved the equation (2) correctly.) So the 25th percentile is 1.

The median \(m\) is found by solving the equation \(F_X(m) = \frac{1}{2}\). Clearly \(m\) must be in \((0, 2)\), so this equation is

\[
\frac{m^2}{4} = \frac{1}{2}.
\]

There are two solutions, namely \(m = \pm \sqrt{2}\). We reject \(-\sqrt{2}\) since it isn’t in \((0, 2)\). (I must see you do this, since otherwise I don’t know you solved the equation (3) correctly.) So the median is \(\sqrt{2}\).  \[\square\]
It is a waste of time to calculate \( f_X(x) = F'_X(x) \) and then solve the equations
\[
\int_{-\infty}^{p} f_X(x) \, dx = \frac{1}{4} \quad \text{and} \quad \int_{-\infty}^{m} f_X(x) \, dx = \frac{1}{2}.
\]

**Problem 8** (Problem 7.18 in the book). Let \( X \) be the continuous random variable with probability density function
\[
f_X(x) = \begin{cases} 
|x| & -1 \leq x \leq 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Find \( \mu_X, E(X^2) \), and \( \sigma_X \).

**Solution.** We have
\[
\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx
= \int_{-1}^{0} x(-x) \, dx + \int_{0}^{1} x \cdot x \, dx
= \left( -\frac{x^3}{3} \right) \bigg|_{-1}^{0} + \frac{x^3}{3} \bigg|_{0}^{1}
= 0 - \left( -\frac{1}{3} \right) + \frac{1}{3} - 0 = 0.
\]

We further have
\[
E(X^2) = \int_{-1}^{0} x^2(-x) \, dx + \int_{0}^{1} x^2 \cdot x \, dx
= \left( -\frac{x^4}{4} \right) \bigg|_{-1}^{0} + \frac{x^4}{4} \bigg|_{0}^{1}
= 0 - \left( -\frac{1}{4} \right) + \frac{1}{4} - 0 = \frac{1}{2}.
\]

Therefore
\[
\sigma_X = \sqrt{E(X^2) - E(X)^2} = \sqrt{\frac{1}{2} - \frac{1}{2}} = \frac{1}{\sqrt{2}}.
\]
(This is approximately 0.70711.)

**Problem 9** (Problem 7.23 in the book). Let \( B \) be the continuous random variable with probability density function
\[
f_B(x) = \begin{cases} 
2x & 0 < x < 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Find \( \mu_B \) and \( \sigma_B \).

**Solution.** We have
\[
\mu_X = E(X) = \int_{0}^{1} x \cdot 2x \, dx = \frac{2x^3}{3} \bigg|_{0}^{1} = \frac{2}{3}.
\]
We further have

\[
E(X^2) = \int_0^1 x^2 \cdot 2x \, dx = \left. \frac{x^4}{2} \right|_0^1 = \frac{1}{2}.
\]

Therefore

\[
\sigma_X = \sqrt{E(X^2) - E(X)^2} = \sqrt{\frac{1}{2} - \left(\frac{2}{3}\right)^2} = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}}.
\]

(This is approximately 0.13357.)

**Problem 10** (Problem 7.26 in the book). Adult American women’s weights have a mean of 145 pounds and a standard deviation of 30 pounds. (The statement of the problem in the book omitted “adult”, without which the claim is certainly false.)

For each of the following weight ranges, what does the rule of thumb (for normal distributions) say about women’s weights in that range? What does Chebyshev’s Inequality say?

1. Between 115 and 175 pounds.
2. Between 85 and 205 pounds.

**Solution.** Recall Chebyshev’s Inequality, for a nonstandardized random variable \( X \) with mean \( \mu_X \) and standard deviation \( \sigma_X \): for any \( r > 0 \),

\[
\Pr(-r\sigma_X < X - \mu_X < r\sigma_X) > 1 - \frac{1}{r^2}.
\]

In this problem, take \( X \) to be the weight in pounds of a random adult woman, so \( \mu_X = 145 \) and \( \sigma_X = 15 \).

**Solution to (1):** We are looking at the interval \( (\mu_X - \sigma_X, \mu_X + \sigma_X) \). So the rule of thumb says about 68% of adult American women should have weights between 115 and 175 pounds. (Using the table inside the front cover of the book, I got the better approximation: the fraction of adult American women with weights between 115 and 175 pounds should be about 0.68468.)

For Chebyshev’s Inequality, take \( r = 1 \) in (4). It says

\[
\Pr(\sigma_X < X - \mu_X < \sigma_X) > 0.
\]

So there are at least some adult American women with weights between 115 and 175 pounds.

**Solution to (2):** We are looking at the interval \( (\mu_X - 2\sigma_X, \mu_X + 2\sigma_X) \). So the rule of thumb says about 95% of adult American women should have weights between 85 and 205 pounds. (Using the table inside the front cover of the book, I got the better approximation: the fraction of adult American women with weights between 85 and 205 pounds should be about 0.95450.)

For Chebyshev’s Inequality, take \( r = 2 \) in (4). It says

\[
\Pr(2\sigma_X < X - \mu_X < 2\sigma_X) > \frac{3}{4}.
\]

So more than 75% of adult American women have weights between 85 and 205 pounds.

**Solution to (3):** We are looking at the interval \( (\mu_X - 3\sigma_X, \mu_X + 3\sigma_X) \). So the rule of thumb says about 99.7% of adult American women should have weights
between 55 and 235 pounds. (The table inside the front cover of the book gives nothing better.)

For Chebyshev’s Inequality, take \( r = 3 \) in (4). It says

\[
\Pr(3\sigma_X < X - \mu_X < 3\sigma_X) > \frac{8}{9}.
\]

So more than about 88.89% of adult American women have weights between 55 and 235 pounds.

\[ \square \]

**Problem 11** (Problem 7.40 in the book). For a 2 year old laptop computer, suppose the total number of years it will be in service is a continuous random variable \( X \) with cumulative distribution function \( F_X \) given by

\[
F_X(x) = \begin{cases} 
1 - \frac{4}{x^2} & x > 2 \\
0 & \text{otherwise} 
\end{cases}
\]

(1) Find an equation for the probability density function.
(2) Find the probability that a 2 year old laptop computer will be in service a total of less than 5 years.
(3) Find the expected value \( \mu_X \).

**Solution.** Solution to (1): Differentiate the function \( F_X \). We can assign an arbitrary value at the point \( x = 0 \). Taking this value to be zero, we get

\[
f_X(x) = F_X'(x) = \begin{cases} 
0 & x \leq 2 \\
8x^{-3} & 2 < x.
\end{cases}
\]

Solution to (2): This is

\[
\Pr(X < 5) = F_X(5) = 1 - \frac{4}{25} = \frac{21}{25} = 0.84.
\]

It is a waste of time to evaluate this as \( \int_{-\infty}^{5} 8x^{-3} \, dx \). You are already given this integral, in the statement of the problem.

Solution to (3): We have

\[
\mu_X = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{2}^{\infty} 8x^{-2} \, dx = (-8x^{-1}) \bigg|_{2}^{\infty} = 0 - \left( \frac{8}{2} \right) = 4.
\]

The units (required) are years. \[ \square \]