Example from last time: John Doe got a score of 3 on a test with scores distributed like \( f(x) = e^{-x} \) for \( x > 0 \), and \( m = 2, \sigma = \frac{1}{3} \).

Jane Wang got a score of \( 2.75 = \frac{4}{2} \) on a test with scores uniformly distributed in \([-1.3, 1.3]\), \( m = 1, \sigma = 1 \).

Who did better?

\[ Z \text{-scores, for John: } \frac{3 - 1}{2} = 2 \text{ mean} \]
\[ \text{for Jane } \frac{2 - 2}{\frac{2}{\sqrt{3}}} = \frac{3}{4} \sqrt{3} \approx 1.299 \]

To John is more 3 \( \sigma \) deviations above the mean than Jane is.

The \( Z \)-scores say much more if distributions are normal (or at least appear normal).

\[ y = f_Z (z) \text{. Total area } = 1 \]

"Bell curve": \( \Phi (z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \, dt \).

Here is a std. notation \( \Phi \), and \( \Phi (z) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right) \).

If \( X \) is normally distributed (with mean \( \mu \), std. dev. \( \sigma \)), then given \( X \), \( Z = \frac{X - \mu}{\sigma} \) give

\[ \Pr (X \leq x) = \Pr (Z \leq z) = \Phi (z) \text{. (Needs } X \text{ normal).} \]

\[ \Phi (-2) \approx 0.0228 \text{ (table inside front cover of book; can use calculators).} \]

\[ \text{green area is also about } 0.0228 \]

\[ \Pr (X \leq \mu + 2\sigma) \approx 0.0228 \text{ (} \approx \Pr (Z \leq -2) \text{).} \]

\[ \Pr (\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9544 \]

Rule of thumb: For roughly normal distribution, 95\% of observations are within 2\( \sigma \) of \( \mu \).

What should \( z \) be to give exactly 95\% of observations between \( \mu - 2\sigma \) and \( \mu + 2\sigma \)?
Find $z$ such that $\Phi(z) = 0.025$ rather than 0.0228.

From table, get $z = 1.96$; so $\Phi(1.96) \approx 0.97500$ correct to digits shown.

Ex: Suppose Math SAT score are normally distributed with $\mu = 500$, $\sigma = 100$.

ACT (same population) $\mu = 18$, $\sigma = 6$.

John Doe got 640 on Math SAT; Jane Wray got 27 on Math ACT.

Find prop. proportion of people getting Math SAT scores < John Doe's score. Need $\Pr(S < 640)$. Use $z = \frac{640 - 500}{100} = 1.4$. Table tells me $\Pr(Z < 1.4) \approx 0.91924$.

\[ \text{Correction here: had 660 before.} \]

For Jane Wray need $\Pr(A < 27)$. Use $z = \frac{27 - 18}{6} = \frac{3}{6} = 0.5$.

By norm, $\Pr(A < 27) > \Pr(S < 640)$. Got $\Pr(A < 27) \approx 0.69149$.

Ex: A philanthropist in the town of Mudville promises a scholarship to anyone getting a Math SAT score so large as to be in the top 0.1% of test takers. How big does score need to be?

Find needed $z$-score. Want $\Phi(z) = 0.999$. This gives $\Pr(Z < z) = 0.999$. From table got $z(3.09) \approx 0.99900$. That corresponds to SAT score: $\mu + z\sigma \approx 500 + (3.09)(100) = 809$.

Note Math SAT scores are only approx normally dist. and I think mean $> 800$. So the philanthropist never has to pay up.

Let $K$ be a binomial RV. $p$: prob of success, $n$: trials. For large $n$. Recall

\[ \mu = np, \quad \sigma = \sqrt{np(1-p)}. \quad \text{For fixed } p, \text{ as } n \to \infty, \text{ get int close (in a suitable sense) to } Z \text{ RV.} \]

Each roosters has burt length $L$ and ave $\Pr(K_n = k)$, with $\frac{np}{k}$ total of best at $k$.

\[ \text{Area } = \binom{n}{k} \pi^k (1-\pi)^{n-k}. \]

Figure 8.5: Probability histograms for binomial random variables with $\pi = \frac{1}{3}$ and $n$ increasing from 3 to 27 to 243.
Example 8.2 Code errors. Radio signals transmitted to earth from deep space probes are very weak against a noisy background. Suppose there is a 3% chance a transmitted bit will be misread. If 1000 bits are transmitted, what is the chance that no more than 35 of the bits were misread?

\[
\text{Binomial distr. } K, \quad \pi = 0.03, \quad n = 1000. \quad \text{Want to know } \Pr(K \leq 35).
\]

Approx. with normal distr. with \( \mu = 1000 \times 0.03 = 30 \), \( \sigma = \sqrt{1000 \times 0.03 \times 0.97} \),

\[
Z \text{-score} = 35 \div 30 = \frac{35 - 30}{\sigma} \approx 0.9688.
\]

Approx.: Use \( \Pr(Z < z) = \Phi(z) \approx 0.8230 \).

Exact answer: \( \sum_{k=0}^{35} \binom{1000}{k} (0.03)^k (0.97)^{1000-k} \approx 0.84608 \).

"Continuity correction" gives a better approx.: \( 0.84608 \) (Misprint in book, calculator).