MATH 413[513] (PHILLIPS) SOLUTIONS TO THE FINAL EXAM

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Book, notes, calculators, and all other electronic devices are prohibited. The only allowed materials are blank paper and pens or pencils.

Write your name and your student ID on your paper.

Total: 200 points, plus extra credit.

1. (10 points/part; total 60 points.) Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required.

(a) Let X be a metric space. Then the countable union of closed subsets of X is closed.

Solution. False. Take $X = \mathbb{R}$, and take $E_n = [0, 1 - \frac{1}{n}]$ for $n \in \mathbb{Z}_{>0}$ with $n \geq 2$. Then $\bigcup_{n=2}^{\infty} E_n = [0, 1)$. This set is not closed because 1 is a limit point which is not in $[0, 1)$.

Details (not required for the solution): The point 1 is a limit point of $[0, 1)$ because if $\varepsilon > 0$, then $x = \max\left(\frac{1}{2}, 1 - \frac{\varepsilon}{2}\right)$ is in ~~$N_{\varepsilon}(1) \cap [0, 1)$~~ $(N_{\varepsilon}(1) \setminus \{1\}) \cap [0, 1)$. □

Alternate solution. False. Take $X = \mathbb{R}$, and take ~~$E_n = \{\frac{1}{n}\}$~~ $E_n = \{\frac{1}{n}\}$ for $n \in \mathbb{Z}_{>0}$. One point sets are closed, so ~~$E = \bigcup_{n=1}^{\infty} E_n = [0, 1)$~~

$$E = \bigcup_{n=1}^{\infty} E_n = \left\{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\right\}$$

is a countable union of closed sets. However, 0 is a limit point of E which is not in E. □

Second alternate solution. False. Take $X = \mathbb{R}$. Then $\mathbb{Q}$ is a countable union of one point sets, so a countable union of closed sets. But $\mathbb{Q}$ is not closed; in fact, $\mathbb{Q}$ is dense. □

(b) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : (a, b) \to \mathbb{C}$ be a function. Let $c \in (a, b)$. If f is continuous at c, then f is differentiable at c.

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Solution. False. Take $a = -1$, $b = 1$, $c = 0$, and $f(x) = |x|$ for $x \in (a, b)$. Then f is continuous at 0, but $f'(0)$ does not exist.

Details (not required for the solution): The function f is among the functions shown to be continuous in Example 4.11 of Rudin's book. (The function there is $x \mapsto \|x\|$ form $\mathbb{R}^d$ to $\mathbb{R}$; this is the special case $d = 1$.)

To show that $f'(0)$ does not exist, observe that

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.$$

Therefore

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. □

(c) Every bounded infinite subset of $\mathbb{R}^2$ has a limit point in $\mathbb{R}^2$.

Solution. True. The closure of a bounded set is still bounded, closed bounded sets in $\mathbb{R}^2$ are compact, and infinite subsets of compact sets have limit points.

Details (not required for the solution): Let $E \subset \mathbb{R}^2$ be a bounded infinite subset. Then there is $r \in [0, \infty)$ such that $E \subset \{x \in \mathbb{R}^2 \colon \|x\| \leq r\}$. The set $\{x \in \mathbb{R}^2 \colon \|x\| \leq r\}$ is closed and bounded. Closed bounded sets of $\mathbb{R}^2$ are compact (Theorem 2.41 of Rudin's book), so $\{x \in \mathbb{R}^2 \colon \|x\| \leq r\}$ is compact. So E is an infinite subset of a compact set, and hence has a limit point by Theorem 2.37 of Rudin's book. □

Alternate solution. The statement is a special case ($k = 2$) of Theorem 2.42 of Rudin's book~~actually has explicitly, as a separate theorem (Theorem 2.42) the statement that Every~~: for $k \in \mathbb{Z}_{>0}$, every bounded infinite subset of ~~$\mathbb{R}^d$~~ $\mathbb{R}^k$ has a limit point in ~~$\mathbb{R}^d$.~~ $\mathbb{R}^k$. □

Second alternate solution. Since E is infinite, there is a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in E whose terms are all distinct. This sequence is bounded, so has a convergent subsequence. The limit of this subsequence is a limit point of E. □

(d) For every $d \in \mathbb{Z}_{>0}$, a nontrivial subset $E \subset \mathbb{R}^d$ is either open or closed.

Solution. False. Take $d = 1$ and $E = [0, 1)$. This set is not closed because 1 is a limit point which is not in $[0, 1)$, and it is not open since $0 \in [0, 1)$ but 0 is not in its interior.

Details (not required for the solution): The point 1 is a limit point of $[0, 1)$ because if $\varepsilon > 0$, then $x = \max\left(\frac{1}{2}, 1 - \frac{\varepsilon}{2}\right)$ is in ~~$N_\varepsilon(1) \cap [0, 1)$~~ $(N_\varepsilon(1) \setminus \{1\}) \cap [0, 1)$.

The point 0 is not in the interior of $[0, 1)$ because if $\varepsilon > 0$, then $-\frac{\varepsilon}{2} \in N_\varepsilon(0)$ but ~~$\frac{\varepsilon}{2} \not\in [0, 1)$~~ $\frac{\varepsilon}{2} \not\in [0, 1)$. ☐

(e) There is metric space X such that every continuous function from X to $\mathbb{R}$ is uniformly continuous.

Solution. True. This is true for $X = [0, 1]$ because $[0, 1]$ is compact and, by Theorem 4.19 of Rudin's book, continuous functions on compact metric spaces are uniformly continuous. ☐

It isn't quite enough to say to take X to be a compact metric space. A complete solution needs to say something about why a compact metric space exists.

Alternate solution. True. This is true for $X = \{0\}$ because *every* function from a one point space to any other metric space is necessarily uniformly continuous. ☐

In fact, any set X with the discrete metric will work, since every function from X to any other metric space is easily seen to be uniformly continuous.

(f) If the real series $\sum_{n=1}^{\infty} a_n$ converges, and $a_n \geq 0$ for all $n \in \mathbb{Z}_{>0}$, then for any subsequence $(a_{k(n)})_{n \in \mathbb{Z}_{>0}}$ the series $\sum_{n=1}^{\infty} a_{k(n)}$ converges.

Solution. True. The n-th partial sum of $\sum_{n=1}^{\infty} a_{k(n)}$ is nonnegative and bounded by the $k(n)$-th partial sum of $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, its partial sums are bounded. So the partial sums of $\sum_{n=1}^{\infty} a_{k(n)}$ are bounded. Since ~~the~~ its terms are nonnegative, this series converges.

Details (not required for the solution): For $n \in \mathbb{Z}_{>0}$, let

$$s_n = \sum_{l=1}^{n} a_l \qquad \text{and} \qquad t_n = \sum_{l=1}^{n} a_{k(l)}.$$

Since $1 \leq k(1) < k(2) < \cdots < k(n)$ and ~~$a_m \leq 0$~~ $a_m \geq 0$ for $m \in \mathbb{Z}_{>0}$, we have

$$0 \leq t_n = \sum_{l=1}^{n} a_{k(l)} \leq \sum_{m=1}^{n\,k(n)} a_m = s_{k(l)\,k(n)} \leq \sum_{m=1}^{\infty} a_m.$$

So $(t_n)_{n \in \mathbb{Z}_{>0}}$ is bounded. Since $a_{k(l)} \geq 0$ for all $n \in \mathbb{Z}_{>0}$, it follows from Theorem 3.24 of Rudin's book that $\sum_{n=1}^{\infty} a_{k(n)}$ converges. ☐

2. (30 points) Let $f \colon [0, 1) \to \mathbb{C}$ be a continuous function. Suppose that $\lim_{x \to 1} f(x)$ exists. Prove that f is bounded.

Solution. Define $g\colon [0,1] \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x) & x \in [0,1) \\ \lim_{t \to 1} f(t) & x = 1. \end{cases}$$

For every $x \in [0,1)$, g is continuous at x because f is continuous at x. Also, g is continuous at 1 because

$$\lim_{t \to 1} g(t) = \lim_{t \to 1} f(t) = g(1).$$

Therefore g is continuous on $[0,1]$. Since $[0,1]$ is compact, g is bounded by Theorem 4.15 of Rudin's book. Therefore f is bounded. □

Alternate solution. Set $L = \lim_{t \to 1} f(t)$. Choose $\delta > 0$ such that for all $t \in [0,1)$ with $|t-1| < \delta$, we have $|f(t) - L| < 1$. Set $r = \max\left(1 - \frac{\delta}{2},\, 0\right)$. Then $r \in [0,1)$. Since $[0,r]$ is compact, Theorem 4.15 of Rudin's book provides $M \in [0, \infty)$ such that $|f(x)| \leq M$ for all $x \in [0,r]$. Also, $[r,1) \subset N_\delta(1) \cap [0,1)$, so for $x \in [r,1)$ we have

$$|f(x)| \leq |L| + |f(x) - L| < |L| + 1.$$

Therefore $|f(x)| \leq \max(M,\, |L|+1)$ for all $x \in [0,1)$. □

3. (20 points) Does the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \frac{1}{2n^2 + \sin(n)}$ converge? Prove your answer. Cite by name any theorems you use (the Comparison Test, etc.), and be sure to verify that their hypotheses are met. You may use the standard facts from elementary calculus about the function $x \mapsto \sin(x)$.

Solution. For $n \in \mathbb{Z}_{>0}$ we have $\sin(n) \geq -1 \geq -n^2$. So $2n^2 + \sin(n) \geq n^2$, whence

$$0 \leq \frac{1}{2n^2 + \sin(n)} \leq \frac{1}{n^2}.$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (Theorem 3.28 of Rudin's book). So the Comparison Test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2 + \sin(n)}$ converges. Since

$$\left| (-1)^{n(n-1)/2} \frac{1}{2n^2 + \sin(n)} \right| = \frac{1}{2n^2 + \sin(n)},$$

the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \frac{1}{2n^2 + \sin(n)}$ converges absolutely. Therefore this series converges. □

The Alternating Series Test does not apply directly. It is true that the absolute values of the terms decrease monotonically to 0. However, the terms don't alternate in sign.

One can group the terms in pairs, getting a series to which the Alternating Series Test does apply. One must then still check that the original series converges. A solution based on this idea is somewhat messy.

4. (30 points) Let $E, F \subset [1, \infty)$ be nonempty and bounded above. Define

$$S = \{ xy \colon x \in E \text{ and } y \in F \}.$$

Prove that $\sup(S)$ exists and is equal to $\sup(E)\sup(F)$.

Solution. Since E and F are nonempty, clearly $S \neq \varnothing$.

We claim that $\sup(E)\sup(F)$ is an upper bound for S. So let $z \in S$. Then there are $x \in E$ and $y \in F$ such that $z = xy$. Since $x, y \geq 0$, we have

$$z = xy \leq \sup(E)\sup(F).$$

The claim is proved.

Since S is nonempty and bounded above, $\sup(S)$ exists.

We claim that if $b < \sup(E)\sup(F)$, then b is not an upper bound for S. Combined with the first claim, this will imply that $\sup(S) = \sup(E)\sup(F)$, and complete the solution.

~~We prove the claim. First, observe that S contains an elements z with $z \geq 0$. If $b < 0$, the claim is therefore trivial. So we can assume $b \geq 0$. Then $\sup(E)\sup(F) > 0$, so $\sup(E) > 0$ and $\sup(F) > 0$.~~

To prove the claim, define

$$\varepsilon = \min\left(\sup(E) 1, \ \sup(F), \frac{\sup(E)\sup(F) - b}{\sup(E) + \sup(F)} \right).$$

Then $\varepsilon > 0$. Therefore there is $x \in E$ such that $x > \sup(E) - \varepsilon$ and there is $y \in F$ such that $y > \sup(F) - \varepsilon$. ~~Now~~ We have $xy \in S$~~and, since~~. Since E is a nonempty subset of $[1, \infty)$, it follows that $\sup(E) \geq 1$. Similarly $\sup(F) \geq 1$. Therefore

$$x \geq 0, y \geq 0 \ \sup(E) - \varepsilon \geq 0, \qquad \text{and} \qquad \sup(F) - \varepsilon \geq 0, 0.$$

~~we have~~ So

$$\begin{aligned} xy &\geq (\sup(E) - \varepsilon)(\sup(F) - \varepsilon) \\ &= \sup(E)\sup(F) - \varepsilon\big(\sup(E) + \sup(F)\big) + \varepsilon^2 \\ &> \sup(E)\sup(F) - \varepsilon\big(\sup(E) + \sup(F)\big) \geq b. \end{aligned}$$

The claim is proved. □

Caution: in this proof, it does *not* follow that, for example, $\sup(E)-\varepsilon \in E$ or $\sup(E) - \frac{\varepsilon}{2} \in E$. All that can be said is that there is $x \in E$ such that $x > \sup(E) - \varepsilon$. Example: take $E = \{1\}$.

5. (30 points) Prove directly from the definition (using an ε-δ argument) that $f(x) = x^2 + x + 1$ is uniformly continuous on $(0, 1)$.

Solution. Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{3}$. Then $\delta > 0$. Now let $x, y \in [0, 1]$ satisfy $|x - y| < \delta$. We clearly have $|x + y| \leq 2$. Therefore

$$\begin{aligned} |f(x) - f(y)| &= \big|(x^2 + x + 1) - (y^2 + y + 1)\big| \\ &\leq \big||x^2 - y^2|\big| + |x - y| \\ &= |x + y| \cdot |x - y| + |x - y| \\ &\leq 2|x - y| + |x - y| \\ &= 3|x - y| < 3\delta = \varepsilon. \end{aligned}$$

This completes the solution. □

6. (30 points) Let X be a metric space, and let A, $B \subset X$. Let $x \in X$ be a limit point of $A \cup B$. Prove that x is a limit point of at least one of the sets A and B.

Solution. Suppose that x is not a limit point of A; we prove that x is a limit point of B.

Since x is not a limit point of A, there is $\varepsilon_0 > 0$ such that ~~$(N_\varepsilon(x) \setminus \{x\}) \cap A = \varnothing$.~~

$$\big(N_{\varepsilon_0}(x) \setminus \{x\}\big) \cap A = \varnothing.$$

Now let $\varepsilon > 0$. Set $\rho = \min(\varepsilon, \varepsilon_0)$. Then ~~$(N_\rho(x) \setminus \{x\}) \cap (A \cup B) \neq \varnothing$. Since $(N_\rho(x) \setminus \{x\}) \cap A = \varnothing$,~~

$$\big(N_\rho(x) \setminus \{x\}\big) \cap (A \cup B) \neq \varnothing.$$

Since

$$\big(N_\rho(x) \setminus \{x\}\big) \cap A = \varnothing,$$

it follows that ~~$(N_\rho(x) \setminus \{x\}) \cap B \neq \varnothing$.~~

$$\big(N_\rho(x) \setminus \{x\}\big) \cap B \neq \varnothing,$$

so

$$\big(N_\varepsilon(x) \setminus \{x\}\big) \cap B \neq \varnothing.$$

We have proved that x is a limit point of B. □

Alternate solution. ~~By definition, there~~ There is a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $(A \cup B) \setminus \{x\}$ such that $\lim_{n \to \infty} x_n = x$. Suppose there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $x_n \in A$. Then there are $k(1) < k(2) < \cdots$ in $\mathbb{Z}_{>0}$ such that $x_{k(n)} \in A$ for all $n \in \mathbb{Z}_{>0}$. So $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ is a sequence in $A \setminus \{x\}$ such that $\lim_{n \to \infty} x_{k(n)} = x$. Thus x is a limit point of A.

Otherwise, there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $x_n \in B$. The same argument as in the previous paragraph shows that there is a subsequence $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ of $(x_n)_{n \in \mathbb{Z}_{>0}}$ which is in $B \setminus \{x\}$. ~~Thus~~ This sequence converges to x, so x is a limit point of B. $\square$

Extra Credit. (30 extra credit points) (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

Let $f \colon [0, \infty) \to \mathbb{R}$ be continuous, and assume that f' exists on $(0, \infty)$. Suppose that $\lim_{x \to \infty} f(x) = f(0)$. Prove or disprove: There is necessarily $c \in (0, \infty)$ such that $f'(c) = 0$.

(For the possible construction of counterexamples, you may use the standard properties of the trigonometric and exponential functions and their inverses from elementary calculus.)

Solution. We prove the statement. If f is constant, there is nothing to prove. Otherwise, there is $x_0 \in (0, \infty)$ such that $f(x_0) > f(0)$, or there is $x_0 \in (0, \infty)$ such that $f(x_0) < f(0)$. We consider only the first case; the second case is handled by applying the first case to $-f$.

Let ~~$\varepsilon = \frac{1}{2}(f(x_0) - f(0)) > 0$~~ $\varepsilon = \frac{1}{2}\big(f(x_0) - f(0)\big) > 0$. Using the definition of $\lim_{x \to \infty} f(x) = f(0)$, we find in particular that there is $r > x_0$ such that $|f(r) - f(0)| < \varepsilon$.

Suppose $f(r) < f(0)$. Since $f(0) < f(x_0)$, the Intermediate Value Theorem provides $s \in (x_0, r)$ such that $f(s) = f(0)$. Applying the Mean Value Theorem on the interval $[0, s]$, we find $x \in (0, s) \subset (0, \infty)$ such that $f'(x) = 0$.

If $f(r) = f(0)$, we apply the Mean Value Theorem on the interval $[0, r]$.

If $f(r) > f(0)$, then the inequality $|f(r) - f(0)| < \frac{1}{2}(f(x_0) - f(0))$ implies that $f(r) < f(x_0)$. The Intermediate Value Theorem now provides $s \in (0, x_0)$ such that $f(s) = f(r)$. Apply the Mean Value Theorem on the interval $[s, r]$ and proceed as before. $\square$