MATH 413[513] (PHILLIPS) MIDTERM: 26 OCT. 2018

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Write your name and your student ID on your paper.

Total: 120 points; time: 120 minutes.

1. (25 points) Let $E, F \subset \mathbb{R}$ be nonempty and bounded above. Define

$$S = \big\{ x + y \colon x \in E \text{ and } y \in F \big\}.$$

Prove that $\sup(S)$ exists and is equal to $\sup(E) + \sup(F)$.

2. (25 points) Let $d \in \mathbb{Z}_{>0}$ and let $U \subset \mathbb{C}^d$ be an open set. Prove carefully that every point $x \in U$ is a limit point of U. (Be as explicit as possible in your proof.)

3. (20 points) Give, with proof, an example of a metric space X with metric d, a number $r > 0$, and a point $x_0 \in X$, such that

$$\overline{N_r(x_0)} \neq \big\{ x \in X \colon d(x, x_0) \leq r \big\}.$$

(Hint: One possible approach [not the fastest] is to take X to be a suitable subset of $\mathbb{R}$.)

4. (25 points) Let X be a metric space, let $x \in X$, and let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in X such that $\lim_{n \to \infty} x_n = x$. Let

$$K = \{x_1, x_2, x_3, \ldots\} \cup \{x\} \subset X.$$

Prove that K is compact.

5. (25 points) Define a sequence $(b_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ by

$$b_n = \begin{cases} n & n \text{ is divisible by } 41 \\ \frac{1}{n} & n \text{ is not divisible by } 41. \end{cases}$$

Find, with proof, all subsequential limits of the sequence $(b_n)_{n \in \mathbb{Z}_{>0}}$. (In particular, for every real number x which is not a subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$, you must prove this fact.)

Extra Credit. (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 75% on the rest of the exam.)

Give, with proof, an example of a complete metric space X with metric d and a bounded sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in X which has no convergent subsequence.