MATH 413[513] (PHILLIPS) SOLUTIONS TO MIDTERM (EXPANDED AND CORRECTED)

Warning: essentially no proofreading of the new parts has been done.

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Write your name and your student ID on your paper.

Total: 120 points; time: 120 minutes.

1. (25 points) Let $E, F \subset \mathbb{R}$ be nonempty and bounded above. Define

$$S = \big\{x + y \colon x \in E \text{ and } y \in F\big\}.$$

Prove that $\sup(S)$ exists and is equal to $\sup(E) + \sup(F)$.

Solution. Since E and F are nonempty, clearly $S \neq \varnothing$.

We claim that $\sup(E) + \sup(F)$ is an upper bound for S. So let $z \in S$. Then there are $x \in E$ and $y \in F$ such that $z = x + y$. So

$$z = x + y \leq \sup(E) + \sup(F).$$

The claim is proved.

Since S is nonempty and bounded above, $\sup(S)$ exists.

We claim that if $b < \sup(E) + \sup(F)$, then b is not an upper bound for S. Combined with the first claim, this will imply that $\sup(S) = \sup(E) + \sup(F)$, and complete the solution.

To prove the claim, define

$$\varepsilon = \frac{\sup(E) + \sup(F) - b}{2}.$$

Then $\varepsilon > 0$. Therefore there is $x \in E$ such that $x > \sup(E) - \varepsilon$ and there is $y \in F$ such that $y > \sup(F) - \varepsilon$. Now $x + y \in S$ and

$$x + y > (\sup(E) - \varepsilon) + (\sup(F) - \varepsilon) = \sup(E) + \sup(F) - 2\varepsilon = b.$$

The claim is proved. □

Caution: in this proof, it does *not* follow that, for example, $\sup(E) - \varepsilon \in E$ or $\sup(E) - \frac{\varepsilon}{2} \in E$. All that can be said is that there is $x \in E$ such that $x > \sup(E) - \varepsilon$. Example: take $E = \{0\}$.

Alternate solution. Prove that $S \neq \varnothing$ and that $\sup(E) + \sup(F)$ is an upper bound for S in the same way these are done in the first solution. In particular, $\sup(S)$ exists.

We claim that $\sup(E) + \sup(F) \leq \sup(S)$. This will finish the solution. To prove the claim, temporarily fix $y \in F$. For all $x \in E$, we have $x + y \in S$, so

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$x+y \leq \sup(S)$. Therefore $x \leq \sup(S) - y$. Since this is true for all $x \in E$, it follows that $\sup(E) \leq \sup(S) - y$. Therefore

$$y \leq \sup(S) - \sup(E).$$

This equation is valid for all $y \in F$. Therefore $\sup(F) \leq \sup(S) - \sup(E)$. Rearranging gives $\sup(E) + \sup(F) \leq \sup(S)$. The claim is proved. □

2. (25 points) Let $d \in \mathbb{Z}_{>0}$ and let $U \subset \mathbb{C}^d$ be an open set. Prove carefully that every point $x \in U$ is a limit point of U. (Be as explicit as possible in your proof.)

Solution. Let $x = (x_1, x_2, \ldots, x_d) \in U$. Since U is open, there is $r > 0$ such that $N_r(x) \subset U$. To show that x is a limit point of U, let $\varepsilon > 0$ be arbitrary. We exhibit a point $y \in (N_\varepsilon(x) \setminus \{x\}) \cap U$. Take $s = \frac{1}{2}\min(r, \varepsilon)$. Then $s > 0$. Define $y = (x_1 + s,\, x_2,\, x_3,\, \ldots,\, x_d)$. Then

$$y \in (N_\varepsilon(x) \setminus \{x\}) \cap N_r(x) \subset (N_\varepsilon(x) \setminus \{x\}) \cap U.$$

This completes the solution. □

The problem said to be explicit. Thus, a full solution requires explicitly giving a point in $(N_\varepsilon(x) \setminus \{x\}) \cap U$, or an explicit proof that there is one. Part of the point is that $\mathbb{C}^d$ has special properties; the result isn't true in a general metric space.

3. (20 points) Give, with proof, an example of a metric space X with metric d, a number $r > 0$, and a point $x_0 \in X$, such that

$$\overline{N_r(x_0)} \neq \big\{x \in X \colon d(x, x_0) \leq r\big\}.$$

(Hint: One possible approach [not the fastest] is to take X to be a suitable subset of $\mathbb{R}$.)

Solution. Set $X = \{0, 1\} \subset \mathbb{R}$, with the subspace metric, and set $r = 1$. Set $x_0 = 0$. Then $N_r(x_0) = \{0\}$. Since this set is finite, it has no limit points. So $N_r(x_0)$ is closed. Therefore $\overline{N_r(x_0)} = \{0\}$. However,

$$1 \in \big\{x \in X \colon d(x, x_0) \leq r\big\}.$$

This completes the solution. □

Alternate solution. Take X to be any set with at least two elements. Define $d \colon X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

We claim that (X, d) is a metric space. Let $x, y \in X$. The conditions $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(y, x) = d(x, y)$ are all obvious. For the triangle inequality, let $x, y, z \in X$. Unless $x = y = z$, we have

$$d(x, y) + d(y, z) \geq 1 \geq d(x, z).$$

If $x = y = z$ then

$$d(x, y) + d(y, z) = 0 = d(x, z).$$

The claim is proved.

Now let $x_0 \in X$ be arbitrary, and set $r = 1$. Then

$$N_r(x_0) = \{x_0\} \qquad \text{and} \qquad \big\{x \in X \colon d(x, x_0) \leq r\big\} = X.$$

The set $N_r(x_0)$ is closed because, being finite, it has no limit points. So $\overline{N_r(x_0)} = \{x_0\}$. Since X has more than one point, it follows that $\overline{N_r(x_0)} \neq X$. $\square$

Second alternate solution. This solution differs from the alternate solution only in the method used to show that $N_r(x_0)$ is closed. We observe that

$$X \setminus N_r(x_0) = X \setminus \{x_0\} = \bigcup_{x \in X \setminus \{x_0\}} N_{1/2}(x).$$

For every $x \in X$, the set $N_{1/2}(x)$ is open. So $X \setminus N_r(x_0)$ is open, whence $N_r(x_0)$ is closed. $\square$

Third alternate solution. Set $X = (-\infty, 0] \cup \{1\} \subset \mathbb{R}$, with the subspace metric, and set $r = 1$. Set $x_0 = 0$. Then $N_r(x_0) = (-1, 0]$. Since $[-1, 0]$ is a closed subset of $\mathbb{R}$, all limit points of $(-1, 0]$ in $\mathbb{R}$ are in $[-1, 0]$. Therefore all limit points of $(-1, 0]$ in X are in $[-1, 0]$. Thus $\overline{N_r(x_0)} \subset [-1, 0]$. In particular, $1 \notin \overline{N_r(x_0)}$. However,

$$1 \in \big\{ x \in X \colon d(x, x_0) \leq r \big\}.$$

This completes the solution. $\square$

Fourth alternate solution. Set $X = \mathbb{R}$. Define $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = \min(|x - y|, \, 1)$$

for $x, y \in \mathbb{R}$.

We claim that d is a metric. For $x, y \in \mathbb{R}$, the conditions $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(y, x) = d(x, y)$ are all immediate. For the triangle inequality, let $x, y, z \in \mathbb{R}$. If $|x - y| \geq 1$ or $|y - z| \geq 1$,

$$d(x, y) + d(y, z) \geq 1 \geq d(x, z).$$

If $|x - y| < 1$ and $|y - z| < 1$ but $|x - y| + |y - z| \geq 1$, then

$$d(x, y) + d(y, z) = |x - y| + |y - z| \geq 1 \geq d(x, z).$$

The only remaining case is $|x - y| + |y - z| < 1$. Then $|y - z| < 1$, so

$$d(x, y) + d(y, z) = |x - y| + |y - z| \geq |y - z| = d(x, z).$$

The claim is proved.

Now set $r = 1$ and $x_0 = 0$. Then

$$N_r(x_0) = (-1, 1) \qquad \text{and} \qquad \big\{ x \in X \colon d(x, x_0) \leq r \big\} = \mathbb{R}.$$

We claim that $2 \notin \overline{N_r(x_0)}$. Obviously $2 \notin N_r(x_0)$. Also, $N_1(2) = (1, 3)$, so $N_1(2) \cap N_r(x_0) = \varnothing$. This shows that 2 is not a limit point of $N_r(x_0)$. The claim is proved.

Since $2 \notin \overline{N_r(x_0)}$, we have

$$\overline{N_r(x_0)} \neq \big\{ x \in X \colon d(x, x_0) \leq r \big\}.$$

This completes the solution. $\square$

Taking $X = \mathbb{Q}$, with the metric it gets as a subspace of $\mathbb{R}$, taking $x_0 = 0$, and taking $r = \sqrt{3}$ (or some other irrational number), does *not* work. If $r \in (0, \infty)$ but $r \notin \mathbb{Q}$, then the following sets (with neighborhoods and closures taken relative to $\mathbb{Q}$) are all equal:

$$N_r(0), \quad \overline{N_r(0)}, \quad \big\{ x \in \mathbb{Q} \colon d(x, 0) \leq r \big\}, \quad (-r, r) \cap \mathbb{Q}, \quad \text{and} \quad [-r, r] \cap \mathbb{Q}.$$

4. (25 points) Let X be a metric space, let $x \in X$, and let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in X such that $\lim_{n \to \infty} x_n = x$. Let

$$K = \{x_1, x_2, x_3, \dots\} \cup \{x\} \subset X.$$

Prove that K is compact.

Solution. We verify that every open cover of K has a finite subcover.

Let $(U_i)_{i \in I}$ be an open cover of K. Choose $j \in I$ such that $x \in U_j$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U_j$. Choose $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies $d(x_n, x) < \varepsilon$. Then $n \geq N$ implies $x_n \in U_j$. For $n = 1, 2, \dots, N-1$, choose $i(n) \in I$ such that $x_n \in U_{i(n)}$. Set

$$F = \big\{ i(1),\, i(2),\, \dots,\, i(N-1),\, j \big\} \subset I.$$

Then $(U_i)_{i \in F}$ is a finite subfamily of $(U_i)_{i \in I}$ which covers K. □

It is also correct to say that a set K is compact if and only if whenever $\mathcal{U}$ is a collection of open sets which covers K, then there is a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ which covers K. (The only difference in the formulation is that this version doesn't allow for repetition of the sets. Sometimes it is not convenient to allow repetition.)

Since people have used this formulation of the definition, but gotten the notation wrong, we give the proof using this formulation.

Alternate solution. We verify that every open cover of K has a finite subcover.

Let $\mathcal{U}$ be an open cover of K. Choose $W \in \mathcal{U}$ such that $x \in W$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset W$. Choose $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies $d(x_n, x) < \varepsilon$. Then $n \geq N$ implies $x_n \in W$. For $n = 1, 2, \dots, N-1$, choose $U_n \in \mathcal{U}$ such that $x_n \in U_n$. Set

$$\mathcal{V} = \big\{ U_1,\, U_2,\, \dots,\, U_{N-1},\, W \big\} \subset \mathcal{U}.$$

Then $\mathcal{V}$ is a finite subcollection of $\mathcal{U}$ which covers K. □

5. (25 points) Define a sequence $(b_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ by

$$b_n = \begin{cases} n & n \text{ is divisible by } 41 \\ \frac{1}{n} & n \text{ is not divisible by } 41. \end{cases}$$

Find, with proof, all subsequential limits of the sequence $(b_n)_{n \in \mathbb{Z}_{>0}}$. (In particular, for every real number x which is not a subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$, you must prove this fact.)

Solution. We claim that 0 is a subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$. To prove this, define $k \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ by $k(n) = 41n + 1$. Then for all $n \in \mathbb{Z}_{>0}$ we have

$$b_{k(n)} = \frac{1}{41n + 1}.$$

To show that $\lim_{n \to \infty} b_{k(n)} = 0$, let $\varepsilon > 0$, and choose some $N \in \mathbb{Z}_{>0}$ with $N > \varepsilon^{-1}$. Let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq N$. Then

$$|b_{k(n)} - 0| = \frac{1}{41n + 1} \leq \frac{1}{41N + 1} < \frac{1}{N} < \varepsilon.$$

So $b_{k(n)} \to 0$, and the claim is proved.

Now let $r \in \mathbb{R} \setminus \{0\}$. We claim that r is not a subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$. So let $(b_{k(n)})_{n \in \mathbb{Z}_{>0}}$ be any subsequence of $(b_n)_{n \in \mathbb{Z}_{>0}}$. (That is, let $k \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

be any strictly increasing function.) We show that $b_{k(n)} \not\to r$. Define $\varepsilon = |r|/2$. Then $\varepsilon > 0$. Let $N \in \mathbb{Z}_{>0}$. Choose $n \in \mathbb{Z}_{>0}$ with

$$n > \max(N,\, 2|r|,\, 2|r|^{-1}).$$

Then $k(n) \geq n$. If $k(n)$ is divisible by 41, then

$$|b_{k(n)} - r| = |k(n) - r| \geq k(n) - |r| \geq n - |r| > 2|r| - |r| > \varepsilon.$$

If $k(n)$ is not divisible by 41, then

$$|b_{k(n)} - r| = \left| \frac{1}{k(n)} - r \right| \geq |r| - \frac{1}{k(n)} \geq |r| - \frac{1}{n} > |r| - \frac{1}{2|r|^{-1}} = \frac{|r|}{2} = \varepsilon.$$

This completes the proof that $b_{k(n)} \not\to r$, and thus proves the claim.

Thus 0 is the only subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$. □

If you look for all subsequential limits of $(b_n)_{n \in \mathbb{Z}_{>0}}$ in $[-\infty, \infty]$, then ∞ is also a subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$. This wasn't the intent, but a solution giving a correct proof that the subsequential limits in $[-\infty, \infty]$ are exactly 0 and ∞ gets full credit.

Caution: for a general sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in a metric space X, a subsequential limit of $(x_n)_{n \in \mathbb{Z}_{>0}}$ is *not* the same thing as a limit point of $\{x_n \colon n \in \mathbb{Z}_{>0}\}$. Example: take $X = \mathbb{R}$, and let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be the sequence

$$x_n = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$$

for $n \in \mathbb{Z}_{>0}$. Then $\{x_n \colon n \in \mathbb{Z}_{>0}\} = \{0, 1\}$, which is finite and so has no limit points. However, 0 and 1 are both subsequential limits of $(x_n)_{n \in \mathbb{Z}_{>0}}$.

Also, $(\frac{1}{n})_{n \in \mathbb{Z}_{>0}}$ is *not* a subsequence of $(b_n)_{n \in \mathbb{Z}_{>0}}$. For example, this sequence has a term equal to $\frac{1}{41}$, but no term of $(b_n)_{n \in \mathbb{Z}_{>0}}$ is equal to $\frac{1}{41}$.

Finally, if (using traditional subsequence notation), if $(b_{n_k})_{k \in \mathbb{Z}_{>0}}$ is a subsequence of $(b_n)_{n \in \mathbb{Z}_{>0}}$, and you want to prove that $b_{n_k} \to x$, then for all $\varepsilon > 0$ you must find $N \in \mathbb{Z}_{>0}$ such that for every $k \geq n$ one has $|b_{n_k} - x| < \varepsilon$. It is wrong to require instead $n_k \geq N$.

Alternate solution. First verify, as in the first solution, that 0 is a subsequential limit of $(b_n)_{n \in \mathbb{Z}_{>0}}$.

Next, let $(b_{k(n)})_{n \in \mathbb{Z}_{>0}}$ be any subsequence of $(b_n)_{n \in \mathbb{Z}_{>0}}$. (That is, let $k \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be any strictly increasing function.) We claim that if $(b_{k(n)})_{n \in \mathbb{Z}_{>0}}$ converges, then $\lim_{n \to \infty} b_{k(n)} = 0$. Suppose first that there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $k(n)$ is divisible by 41. Then for every $M \in \mathbb{R}$ there is $n > M$ such that $k(n)$ is divisible by 41, so $b_{k(n)} = n > M$. This shows that $(b_{k(n)})_{n \in \mathbb{Z}_{>0}}$ is not bounded, a contradiction. So there are only finitely many $n \in \mathbb{Z}_{>0}$ such that $k(n)$ is divisible by 41. Choose $N_1 \in \mathbb{Z}_{>0}$ such that for all $n \geq N_1$,' the number $k(n)$ is not divisible by 41. Let $\varepsilon > 0$. Choose $N_2 \in \mathbb{Z}_{>0}$ such that $1/N_2 < \varepsilon$. Set $N = \max(N_1, N_2)$. Let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq N$. Then $k(n)$ is not divisible by 41 since $n \geq N_1$. Therefore $b_{k(n)} = 1/k(n)$. Now, using $k(n) \geq n$ at the third step and $n \geq N_2$ at the fourth step, we have

$$|b_{k(n)} - 0| = \left| \frac{1}{k(n)} - 0 \right| = \frac{1}{k(n)} \leq \frac{1}{n} \leq \frac{1}{N_2} < \varepsilon.$$

So $\lim_{n \to \infty} b_{k(n)} = 0$. The claim is proved.

The claim shows that no point other than 0 is a subsequential limit of $(b_n)_{n\in\mathbb{Z}_{>0}}$. Therefore the subsequential limits are 0 and nothing else. □

Second alternate solution. First verify, as in the first solution, that 0 is a subsequential limit of $(b_n)_{n\in\mathbb{Z}_{>0}}$.

Next, let $(b_{k(n)})_{n\in\mathbb{Z}_{>0}}$ be any subsequence of $(b_n)_{n\in\mathbb{Z}_{>0}}$. (That is, let $k\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be any strictly increasing function.) We claim that if $(b_{k(n)})_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence, then there are only finitely many $n \in \mathbb{Z}_{>0}$ such that $k(n)$ is divisible by 41. We prove the claim by contradiction. So suppose there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $k(n)$ is divisible by 41. Set $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{Z}_{>0}$. Choose $n \geq N$ such that $k(n)$ is divisible by 41. Choose $m > n$ such that $k(m)$ is divisible by 41. Then, since $k(m)$ and $k(n)$ are distinct integers, we have

$$|b_{k(m)} - b_{k(n)}| = |k(m) - k(n)| \geq 1 > \varepsilon.$$

The claim is proved.

Since convergent sequences are Cauchy, it follows that if $(b_{k(n)})_{n\in\mathbb{Z}_{>0}}$ converges, then there are only finitely many $n \in \mathbb{Z}_{>0}$ such that $k(n)$ is divisible by 41. Now prove, as in the second solution, that $\lim_{n\to\infty} b_{k(n)} = 0$. Thus no point other than 0 is a subsequential limit of $(b_n)_{n\in\mathbb{Z}_{>0}}$. Therefore the subsequential limits are 0 and nothing else. □

Extra Credit. (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 75% on the rest of the exam.)

Give, with proof, an example of a complete metric space X with metric d and a bounded sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$ in X which has no convergent subsequence.

Solution. Take X to be any infinite set. Define $d\colon X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

We claim that (X, d) is a metric space. Let $x, y \in X$. The conditions $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(y, x) = d(x, y)$ are all obvious. For the triangle inequality, let $x, y, z \in X$. Unless $x = y = z$, we have

$$d(x, y) + d(y, z) \geq 1 \geq d(x, z).$$

If $x = y = z$ then

$$d(x, y) + d(y, z) = 0 = d(x, z).$$

The claim is proved.

We claim that every Cauchy sequence in X is eventually constant, that is, that if a sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$ in X is Cauchy, then there are $N \in \mathbb{Z}_{>0}$ and $b \in X$ such that for all $n \geq N$ we have $x_n = b$. So let $(x_n)_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in X. Choose $N \in \mathbb{Z}_{>0}$ such that for all $m, n \in \mathbb{Z}_{>0}$ with $m, n \geq N$ we have $d(x_m, x_n) < \frac{1}{2}$. Set $b = x_N$. For $n \geq N$, we have $d(x_b, b) < \frac{1}{2}$. The definition of d implies that $x_n = b$. The claim is proved.

Since eventually constant sequences converge, it is now immediate that X is complete.

Since every Cauchy sequence in X is eventually constant and every convergent sequence is Cauchy, it is immediate that every convergent sequence in X is eventually constant.

Use infiniteness of X to choose a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in X whose terms are all distinct. Then the terms of every subsequence of $(x_n)_{n \in \mathbb{Z}_{>0}}$ are also all distinct. Therefore $(x_n)_{n \in \mathbb{Z}_{>0}}$ has no convergent subsequence.

The sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ is bounded because $d(x_n, x_1) \leq 1$ for all $n \in \mathbb{Z}_{>0}$. □

Here are two further solutions, which are important but which use ideas not yet covered in the course, so that I don't expect people to have found them.

Alternate solution (sketch). Let X be the set of all bounded functions $\xi \colon \mathbb{Z}_{>0} \to \mathbb{C}$. (Thus, X is the set of all bounded sequences in $\mathbb{C}$, but with elements written in function notation.) Define $d(\xi, \eta) = \sup_{k \in \mathbb{Z}_{>0}} |\xi(k) - \eta(k)|$. This makes X a metric space (easy), and X is complete in this metric (less easy, but not really hard).

Now define a sequence $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ in X by

$$\xi_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

For $m \neq n$, we have $d(\xi_m, \xi_n) = 1$, from which it is easy to see that $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ has no convergent subsequence.

The sequence $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ is bounded because for all $n \in \mathbb{Z}_{>0}$ we have $d(\xi_1, \xi_n) \leq 1$. □

The subspace metric on the set $\{\xi_n \colon n \in \mathbb{Z}_{>0}\}$ is exactly the metric of the first solution.

Second alternate solution. Set $X = \mathbb{R}$. Define $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = \min(|x - y|, \, 1)$$

for $x, y \in \mathbb{R}$.

We claim that d is a metric. For $x, y \in \mathbb{R}$, the conditions $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(y, x) = d(x, y)$ are all immediate. For the triangle inequality, let $x, y, z \in \mathbb{R}$. If $|x - y| \geq 1$ or $|y - z| \geq 1$,

$$d(x, y) + d(y, z) \geq 1 \geq d(x, z).$$

If $|x - y| < 1$ and $|y - z| < 1$ but $|x - y| + |y - z| \geq 1$, then

$$d(x, y) + d(y, z) = |x - y| + |y - z| \geq 1 \geq d(x, z).$$

The only remaining case is $|x - y| + |y - z| < 1$. Then $|y - z| < 1$, so

$$d(x, y) + d(y, z) = |x - y| + |y - z| \geq |y - z| = d(x, z).$$

The claim is proved.

We claim that a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ converges to x with respect the the metric d if and only if it converges to x with respect the the usual metric on $\mathbb{R}$. To prove the claim, first assume that $(x_n)_{n \in \mathbb{Z}_{>0}}$ converges to x with respect the the usual metric on $\mathbb{R}$. Let $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{>0}$ such that for all $n \in \mathbb{Z}_{>0}$ with $n \geq N$, we have $|x_n - x| < \varepsilon$. For $n \in \mathbb{Z}_{>0}$ with $n \geq N$, we then have

$$d(x_n, x) \leq |x_n - x| < \varepsilon.$$

So $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ converges to x with respect the the metric d. Conversely, suppose $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ converges to x with respect the the metric d. Let $\varepsilon > 0$. Set $\varepsilon_0 = \min(\varepsilon, \frac{1}{2})$. Choose $N \in \mathbb{Z}_{>0}$ such that for all $n \in \mathbb{Z}_{>0}$ with $n \geq N$, we have $d(x_n, x) < \varepsilon_0$. Let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq N$. Since $d(x_n, x) < 1$, we must have

$d(x_n, x) = |x_n - x|$. Therefore $|x_n - x| < \varepsilon$. So $(x_n)_{n \in \mathbb{Z}_{>0}}$ converges to x with respect the the usual metric on $\mathbb{R}$. The claim is proved.

A very similar argument shows that sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ is Cauchy with respect the the metric d if and only if is Cauchy with respect the the usual metric on $\mathbb{R}$.

We claim that X is complete with respect the the metric d. To prove the claim, let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ which is Cauchy with respect the the metric d. By the previous paragraph, $(x_n)_{n \in \mathbb{Z}_{>0}}$ is is Cauchy with respect the the usual metric on $\mathbb{R}$. Since $\mathbb{R}$ is complete in the the usual metric, there is $x \in \mathbb{R}$ such that $(x_n)_{n \in \mathbb{Z}_{>0}}$ converges to x with respect the the usual metric. By the previous claim, $(x_n)_{n \in \mathbb{Z}_{>0}}$ converges to x with respect to d. The claim is proved.

Now define a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in X by $x_n = n$ for all $n \in \mathbb{Z}_{>0}$. Every subsequence of $(x_n)_{n \in \mathbb{Z}_{>0}}$ is unbounded. Therefore $(x_n)_{n \in \mathbb{Z}_{>0}}$ has no subsequence which converges in the the usual metric. By the first claim, $(x_n)_{n \in \mathbb{Z}_{>0}}$ has no subsequence which converges with respect to d. However, $(x_n)_{n \in \mathbb{Z}_{>0}}$ is bounded because $d(x_n, 0) = 1$ for all $n \in \mathbb{Z}_{>0}$. □

Let $h \colon \mathbb{R} \to \mathbb{R}$ be the identity function, regarded as a map from $\mathbb{R}$ with the usual metric to $\mathbb{R}$ with the metric d. The fact that d has the same convergent sequences as the usual metric is the statement that h and h^{-1} are continuous. The fact that d has the same Cauchy sequences as the usual metric is the statement that h and h^{-1} are uniformly continuous.

Third alternate solution (sketch). Take $X = \mathbb{R}$, Define $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

for $x, y \in \mathbb{R}$.

First, prove that d is a metric. (This was in a homework problem.) Then prove that d has the same convergent sequences and the same Cauchy sequences as the usual metric. These are similar to, but more complicated than, the proofs of the corresponding statements in the second alternate solution.

As in the second alternate solution, it follows that $\mathbb{R}$ is complete with respect the the metric d.

Given this, the same sequence as in the second alternate solution also works here. □