MATHEMATICS 413 [513]: FALL QUARTER, 2018

Corrected and updated 10 November 2018.

Course Number: Math 413 and Math 513.

Course Title: Introduction to Analysis 1.

Class Time and Place: MWF 11–11:50 am in 260 Condon. Discussion section Tu 11–11:50 am in 104 Deady.

I will be away at conferences 22–26 October (5th week of classes) and 26–30 November (10th week of classes). I will arrange makeup lectures if possible. More likely, the lectures will be given by substitutes.

Prerequisites: Math 282 and Math 317. Math 263 can substitute for Math 317.

CRN: 13786 (Math 413); 13794 (Math 513).

Instructor: N. Christopher Phillips.

Office: 105 Deady.

Office hours: MF 3:00–3:50 pm, Th 10–10:50 am, **or by appointment**. (Office hours are subject to change.)

Please knock. I have found it necessary to keep my office door closed, usually even during office hours, because of my location. Otherwise I get asked all the time where to find nonexistent rooms or the Math Department office.

Email: See my home page or the course website. The subject line of your message should start with "M413", followed by your last name, then first initial.

I might use email occasionally to distribute general announcements. I will give short replies to emailed questions. I don't type at a reasonable speed, so I will rarely answer complicated questions by email. Please come to office hours instead.

When emailing me, please use plain text (7 bit ASCII) only. That is, only the characters found on a standard English language keyboard; no curved quotation marks, curved apostrophes, accented letters, Greek letters, etc. Use TEX pseudocode for mathematical symbols, including Greek letters. Don't send html; for security and privacy reasons, I don't read html email. Don't use MIME or other encoding. Send binary "attachments" only by prior arrangement. Never send Microsoft Word files—I can't read them.

Course Website: `http://pages.uoregon.edu/ncp/Courses/Math413_F18_Web/Math413_F18_Web.html`, or go to my home page (`https://pages.uoregon.edu/ncp/`) and follow the link to Math 413[513].

Copies of some course materials will be posted here.

Course Description: The three quarter sequence Math 413–Math 415 give a rigorous presentation of calculus of one and several variables (the material of Math 251–Math 253 and Math 281–Math 282). In Math 413, we will basic properties of $\mathbb{R}$ and $\mathbb{C}$, elementary point set topology and continuity

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in the context of metric spaces (which includes all subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$), convergence of series, and differentiation.

Textbook: W. Rudin, *Principles of Mathematical Analysis,* 3rd edition.

Notes on Terminology: The book contains a few oddities of notation and terminology which I will not follow. Open intervals are referred to as "segments". The sets of real numbers and rational numbers are called R and Q instead of the more usual $\mathbb{R}$ and $\mathbb{Q}$. (In particular, I will feel free to use the letters R and Q for other purposes.) In contrast to this, the book suddenly switches from f to $\mathbf{f}$ when the function is supposed to take values in $\mathbb{R}^n$ instead of a general metric space; I will make no such distinction.

To avoid the perpetual confusion about whether $0 \in \mathbb{N}$, in anything formal (including homework and solutions) I will take $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$ and $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. These symbols are unfortunately ugly, but they are unambiguous.

To avoid confusion about whether, say, an increasing sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ is supposed to satisfy $x_{n+1} > x_n$ or $x_{n+1} \geq x_n$, in anything formal I will refer only to "nondecreasing" and "strictly increasing" sequences (as well as functions, etc.). Similarly, I will use 'nonincreasing" and "strictly decreasing". To see the problem, compare Rudin's definition of a monotonically increasing sequence with the definition of an increasing sequence in the textbook we are currently using in Math 251–253. For similar reasons, I will avoid "positive" in favor of "nonnegative" and "strictly positive".

Rudin (following a number of authors) defines a set E to be "countable" if there is a bijection from E to $\mathbb{Z}_{>0}$. The other authors also consider finite sets to be countable. (At least one book uses "countable" for one version and "denumerable" for the other.) Rudin's choice is poor: many of the authors (including Rudin) who exclude finite sets in their definition of "countable" nevertheless implicitly include finite sets in practice. One sees this, for example, in Rudin's Problem 2.22. Everybody agrees that a metric space with only one point is separable. But, formally, Rudin's definition excludes such spaces: they don't have countable subsets because they don't have infinite subsets. (A careful author would have written "at most countable".) The wording "a metric space is separable if it has a countable dense subset" (and in similar related definitions) is nearly universal. Therefore I will take "countable" to include finite subsets.

Grading:

Homework: Contribution to final grade: approximately 20%. Due every Monday, except as otherwise stated.

The homework will be hard, some of it quite hard, but it is an essential part of the course.

The homework consists of proofs. This means that everything must be proved. In particular, if a problem asks for an example or counterexample, you must prove that your example has the required properties. If a problem asks whether something is true, you must not only decide whether it is, but also provide a proof or counterexample as appropriate; just as above, counterexamples must be proved to have the required properties.

Proofs should be clear, correct, and complete. They should use complete sentences and correct punctuation. They should have all quantifiers in the right places. They should be in logical order. For example, for a proof of convergence of a sequence from the definition, start with "Let $\varepsilon > 0$" and end with the statement that something is less than ε. If there are subsidiary arguments in a proof, say what is being claimed, say that you are proving it, prove it, and say when the proof of your claim is done. A reader who can't follow any of the mathematics should nevertheless be able to easily tell where the proof of your subsidiary claim starts and where it ends.

I expect to post on the course website at least sketches of some solutions after the homework is turned in.

Some procedural points:

(1) Staple pages together. Don't fold or tear corners or fold in half.
(2) If you cooperate with someone else on any particular problem, you must acknowledge the cooperation (including the name of the person) in your solution.

One Midterm: Contribution to final grade: approximately 30%. Date: Friday 26 Oct. The midterm from the last time I taught the course is on the course website, along with solutions. (The format of the midterm this time may be a bit different.) A large portion of the midterm will ask for proofs which you will have to devise for yourself during the exam period. If I use the format from last time, many people will not finish, and it is likely that getting 75% of it completely right will be a very good score.

I am willing to allow people to start an hour early or continue an hour late, or to arrange to give the entire exam during a longer period outside of the normal class time. Such a procedure can only be followed if suitable rooms and times can be found, and if everybody enrolled in the class agrees to it.

Final Examination: Contribution to final grade: approximately 50%. Date: Thursday 6 Dec., 10:15 am–12:15 pm, in our usual classroom.

No early final exams, according to University rules. If you have another final exam scheduled at the same time as our final exam, you need to give me the details (course number and instructor) by Monday of the 7th week of classes.

Similar comments on the content apply as for the midterm.

I keep originals of the final exam, but will give out on request a scan or copy of your exam.

Accessible Education Center: Students with documented learning disabilities who wish to use the Accessible Education Center (linked on the course website) to take tests under specifically arranged conditions should let me know as soon as possible, certainly by Wednesday of the third week of classes. Such students must **also** be sure to meet the Accessible Education Center's separate deadlines for requests; these are likely to be a week or more in advance of the exam date (much more for final exams). I can't do anything to help a student who misses its deadline. (I have tried in the past.)

Grading Complaints: Complaints about the grading of any exam must be submitted in writing by the beginning of the first class period after the class in which that exam is returned.

Extra Credit: Extra credit will be awarded to the first person to detect any error or misprint in the book, in homework solutions, or in exams and exam solutions. More extra credit will be given for catching mathematical errors than misspelled words or wrong dates. To get the extra credit, you must tell me what the correct version should be.

There are likely to be a number of mistakes to catch. When going through materials from the last time I taught this course, I found several misprints, including one egregious one ("not compact" for "compact"), which were not caught despite a similar offer for extra credit.

The following misprints and mistakes in the book have already been caught, so can't be reused.

- Nonmathematical misprint: Page 4, second paragraph under 1.9: "largest" misspelled.
- Multiple mathematical errors: Rudin defines a set to be countable if it has the same cardinality as $\mathbb{Z}_{>0}$. In problems 2.22–2.26, this gives the wrong definition of the terms "separable" and "countable base". In particular, Problem 2.25 would be false as stated: a one point space is compact, but could not be separable because it wouldn't have any countable subsets.

Learning Outcomes: The course deals with:

- The basic topology of $\mathbb{R}$ and $\mathbb{C}$.
- The topology of metric spaces.
- Convergence of sequences and series in $\mathbb{R}$ and $\mathbb{C}$.
- Continuity of functions and continuous functions between metric spaces.
- Differentiability of functions, differentiable functions, and their derivatives, for functions with domain a suitable subset of $\mathbb{R}$ and codomain $\mathbb{R}$ or $\mathbb{C}$.

The successful student will be able to precisely state the definitions associated with these topics, and will be able to rigorously prove standard facts. The successful student will also be able to rigorously prove or disprove (as appropriate) statements about these topics which have not been encountered before, and which are at a level of difficulty appropriate to a mid-level introductory analysis course. The successful student will be able to write these proofs so that they are correct, complete, clear, readable, and in logical order.

Academic Conduct: The code of student conduct and community standards is linked on the course website. In this course, it is appropriate to help each other on homework as long as the work you are submitting is your own and you understand it, and you give the names of any people you cooperated with. It is not appropriate to help each other while taking exams, to look at other students' exams while taking exams, or to bring any unauthorized material to exams.

Schedule: This schedule is subject to minor changes as may be necessary.

We will cover most of Chapters 1–5 of the book in the fall quarter, and we will follow the book fairly closely.

The book usually used in the preceding course is S. Abbott, *Understanding Analysis.* Much of the material of this quarter is also there, but often restricted to special cases. (As a representative example, continuity is mostly only treated there in the context of functions from subsets of $\mathbb{R}$ to $\mathbb{R}$, and we will consider continuity of functions between metric spaces.) One needs to have had some exposure to this material, at least for motivation; otherwise, the first several weeks of this course will seem hopelessly abstract.

The schedule gives *approximate* times that topics will be discussed. It is necessary to read the book; not everything will be done in class. Homework assignments are **subject to change**.

24–28 Sept.: Chapter 1 (the system of real numbers). Turn in the questionnaire Wednesday 26 September.

Homework (due Monday 1 Oct.): Chapter 1: 1, 2, 5, 6, 9, 13, 17. Point values: 10 points per problem, except Problem 6, which is 10 points for each part, for a total of 40 points..

In Problem 5, it is a mathematical error (which I have seen made before, and which leads to false proofs) to write

~~for all $-x \in -A$.~~

You must consider $x \in -A$, and then say something about $-x$. Note, though, that writing

$$\{-x \colon x \in A\}$$

is perfectly legitimate. (In fact, one of the axioms of set theory explicitly permits this.)

In Problem 6, take as known that the function $n \mapsto b^n$, from $\mathbb{Z}$ to $\mathbb{R}$, is strictly increasing, that is, that for $m, n \in \mathbb{Z}$, we have $b^m < b^n$ if and only if $m < n$. Similarly, take as known that $x \mapsto x^n$ is strictly increasing on $[0, \infty)$ when n is an *integer* with $n > 0$. Further assume that the usual laws of exponents are known to hold when the exponents are *integers*. Don't assume anything about fractional exponents, except for Theorem 1.21 of the book and its corollary, because the context makes it clear that we are to assume fractional powers have not yet been defined.

Part (d) is hard. Be particularly careful that your reasoning is correct. My solution uses instead of $B(x)$ the set

$$B_0(x) = \{b^r : r \in \mathbb{Q} \cap (-\infty, x)\}$$

(that is, we require $r < x$ rather than $r \leq x$) in the definition of b^x. Of course, if you do this you must *prove* that it has the same supremum as $B(x)$.

1–5 Oct.: Start Chapter 2 (metric spaces: basic properties).

Homework (due Monday 8 Oct.): Chapter 2: 2, 3, 4, 5, 6, 8, 9, 11. Point values: 5 points for each part of each problem. For this purpose, Problem 6 is considered to have three parts (it asks you to do three things, even though they are not officially labelled as parts), and similarly Problem 8 is considered to have two parts.

8–12 Oct.: Finish Chapter 2 (more on metric spaces: compactness, etc.).

Homework (due Monday 15 Oct.): Chapter 2: 14, 16, 19, 20, 22, 23, 25.

15–19 Oct.: Start Chapter 3 (sequences).

Homework (due Monday 22 Oct.): Chapter 3: 1, 2, 3, 4, 5, 6, 10, 21, 22.

22–26 Oct.: Continue Chapter 3 (series); review; midterm. (The midterm is Friday 26 Oct.)

Homework (due Monday 29 Oct.; short assignment): Chapter 3: 7, 8.

29 Oct.–2 Nov.: Continue Chapter 3 (series).

Homework (due Monday 5 Nov.): Chapter 3: 9, 16, 23, 24. Also do the following problem.

Problem A. Prove the equivalence of four definitions of the lim sup of a sequence. That is, prove the following theorem.

Theorem. Let $(a_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of $(a_n)_{n \in \mathbb{Z}_{>0}}$ in $[-\infty, \infty]$. Define numbers r, s, t, and $u \in [-\infty, \infty]$ as follows:

(1) $r = \sup(E)$.

(2) $s \in E$ and for every $x > s$, there is $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies $a_n < x$.

(3) $t = \inf_{n \in \mathbb{Z}_{>0}} \sup_{k \geq n} a_k$.

(4) $u = \lim_{n \to \infty} \sup_{k \geq n} a_k$.

Prove that there actually is a unique element $s \in [-\infty, \infty]$ satisfying the condition in (2), that the limit u in (4) exists in $[-\infty, \infty]$, and that $r = s = t = u$.

You do not need to repeat the part that is done in the book (Theorem 3.17).

5–9 Nov.: Finish Chapter 3 (series).

Homework (due Monday 12 Nov.): Chapter 3: 11, 13, 14(a)–(d), 19.

Hint on Problem 14(a): with $s = \lim_{n \to \infty} s_n$, start with

$$|\sigma_n - s| \leq \frac{1}{n+1} \sum_{k=0}^{n} |s_k - s|.$$

Choose $n_0 \in \mathbb{Z}_{>0}$ such that if $n \geq n_0$ then $|s_n - s|$ is small. Then

$$\frac{1}{n+1} \sum_{k=n_0}^{n} |s_k - s|$$

will be small. The sum of the other n_0 terms will be small if $n \gg n_0$.

12–16 Nov.: Start Chapter 4 (continuous functions).

Homework (due Monday 19 Nov.): Chapter 4: 1, 3, 4, 6, 7, 8, 9.

19–21 Nov.: (short week, due to holidays): Finish Chapter 4 (more on continuous functions).

Homework (due **Wednesday** 28 Nov.): Chapter 4: 10, 11, 12, 14, 15, 16, 18.

26 Nov.–30 Nov.: Start Chapter 5 (derivatives); review.

3–7 Dec.: Final exams week.

Tu 4 Dec., 8:00–10:00 pm: Review session **if requested**. Room to be announced.

Th 6 Dec., 10:15 am–12:15 pm: final exam.

7 Jan.–11 Jan. 2019 (first week of Math 414[514]): Finish Chapter 5 (more on derivatives).

Homework (due Monday 14 ~~Nov~~Jan.): Chapter 5: 1, 2, 3, 4, 5, 9, 11, 13.

Important dates: These are not guaranteed, and they are different from previous quarters.

Su 23 Sept.: Last day to process a complete drop (100% refund, no W recorded).

Sa 29 Sept.: Last day to drop this course (100% refund, no W recorded; after this date, W's are recorded).

Sa 29 Sept.: Last day to process a complete drop (90% refund, no W recorded).

Su 30 Sept.: Last day to add this course.

Su 30 Sept.: Last day to withdraw from this course (100% refund, W recorded).

W 3 Oct.: Last day to change to or from audit.

Su 7 Oct.: Last day to withdraw from this course (75% refund, W recorded).

Su 14 Oct.: Last day to withdraw from this course (50% refund, W recorded).

Su 21 Oct.: Last day to withdraw from this course (25% refund, W recorded).

Su 11 Nov.: Last day to withdraw from this course (0% refund, W recorded).

Su 11 Nov.: Last day to change grading option or variable credits for this course. (Tuition penalties apply when reducing credits.)

F 23 Nov.: No classes. (Part of the Thanksgiving break.)