MATH 413[513] (PHILLIPS) FINAL EXAM

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Write your name and your student ID on your paper.

Do you want your grade posted? If so, write a statement to that effect on your exam paper. Specify whether you want the grade posted on the web, outside my door, or both. Also, specify a code under which to post the grade.

Total: 200 points, plus extra credit.

1. (10 points/part; total 60 points.) Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required.

(a) Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$. Suppose that $f'(x) \neq 0$ for all $x \in (a, b)$ for which $f'(x)$ exists. Then there is $x \in (a, b)$ such that $f'(x)$ does not exist.

(b) Define a functions $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

Then $f'(0)$ exists and is equal to zero. (You may use the standard properties of the trigonometric functions from elementary calculus.)

(c) Let $(a_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$, and let $t \in \mathbb{R}$. Suppose $a_n < t$ for all n. Then $\limsup_{n \to \infty} a_n < t$.

(d) Let $E \subset \mathbb{R}$ be a subset which is not compact. Then there exists an unbounded continuous function $f \colon E \to \mathbb{R}$.

(e) Let X and Y be metric spaces, and let $f \colon X \to Y$ be a continuous function. Let $E \subset X$, and let $x_0 \in X$ be a limit point of E. Then $f(x_0)$ is a limit point of $f(E)$.

(f) Let X and Y be metric spaces, and let $f \colon X \to Y$ be a continuous function. Let $E \subset X$, and let $x_0 \in \overline{E}$. Then $f(x_0) \in \overline{f(E)}$.

2. (30 points) Let X be a metric space, and let K and L be compact subsets of X. Prove that $K \cup L$ is compact.

3. (30 points) Define a function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x + 1 & x \neq 0 \\ 2 & x = 0. \end{cases}$$

Prove directly from the definition, with full details, that f is not continuous at 0, but that f is continuous at every other point of $\mathbb{R}$.

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4. (20 points) Prove that there exists $x \in \mathbb{R}$ such that $x^{19} + 7x^3 + 11 = 0$.

5. (30 points) Determine, with proof, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}.$$

6. (30 points) Let $E \subset \mathbb{R}$ be an open interval, and let $f \colon E \to \mathbb{R}$ be differentiable. Suppose that f' is bounded on E. Prove that f is uniformly continuous.

Extra Credit. (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

Prove or disprove the following statement:

"Let $a, b \in \mathbb{R}$ satisfy $a < b$. Let $f, \, g \colon (a, b) \to \mathbb{R}$ be differentiable functions such that

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0,$$

and such that

$$\lim_{x \to 0} \frac{f(x)}{g(x)}$$

exists. Then the equation

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lim_{x \to a^+} \frac{f(x)}{g(x)}$$

holds."

(For the possible construction of counterexamples, you may use the standard properties of the trigonometric and exponential functions and their inverses from elementary calculus.)