MATH 413[513] FINAL EXAM SOLUTIONS FALL 2001

Note: Extra credit will be awarded to the first person who finds any particular error or misprint in these solutions. You must say what the error is and how to correct it. Deadline: Noon Friday 7 Dec. Email notification will be accepted.

1. (10 points/part; total 60 points.) Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required.

(a) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$. Suppose that $f'(x) \neq 0$ for all $x \in (a,b)$ for which $f'(x)$ exists. Then there is $x \in (a,b)$ such that $f'(x)$ does not exist.

Solution. True.

Rolle's Theorem implies that if $f'(x)$ exists for all $x \in (a,b)$, then there is $x \in (a,b)$ such that $f'(x) = 0$. □

(b) Define a functions $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $f'(0)$ exists and is equal to zero. (You may use the standard properties of the trigonometric functions from elementary calculus.)

Solution. True.

By definition, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \left(\frac{1}{h}\right) \cdot h^2 \sin\left(\frac{1}{h^2}\right) = \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right).$$

In this expression,

$$\left| h \sin\left(\frac{1}{h^2}\right) \right| \leq |h|$$

for $h \neq 0$, and $\lim_{h\to 0} |h| = 0$, so

$$\lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right) = 0.$$

Thus $f'(0) = 0$. □

(c) Let $(a_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$, and let $t \in \mathbb{R}$. Suppose $a_n < t$ for all n. Then $\limsup_{n\to\infty} a_n < t$.

Solution. False.

Define $t = 0$ and set $a_n = -\frac{1}{n}$ for $n \in \mathbb{Z}_{>0}$. Then $a_n < t$ for all $n \in \mathbb{Z}_{>0}$. However, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0 = t$. □

Date: 6 Dec. 2001.

(d) Let $E \subset \mathbb{R}$ be a subset which is not compact. Then there exists an unbounded continuous function $f \colon E \to \mathbb{R}$.

Solution. True.

Since closed bounded subsets of $\mathbb{R}$ are compact, E is not bounded or not closed. If E is not bounded, then the function $f \colon E \to \mathbb{R}$ given by $f(x) = x$ for $x \in E$ is continuous but not bounded. So suppose E is not closed. Then there is $x_0 \in \mathbb{R} \setminus E$ which is a limit point of E. Define $f \colon E \to \mathbb{R}$ by $f(x) = (x - x_0)^{-1}$ for $x \in E$. To show f is not bounded, let $M \in (0, \infty)$; we find $x \in E$ such that $|f(x)| > M$. Set $\varepsilon = \frac{1}{M} > 0$. Since x_0 is a limit point of E, there is $x \in E$ such that $|x - x_0| < \varepsilon$. Then

$$|f(x)| = \frac{1}{|x - x_0|} > \frac{1}{\varepsilon} = M.$$

This completes the solution. □

(e) Let X and Y be metric spaces, and let $f \colon X \to Y$ be a continuous function. Let $E \subset X$, and let $x_0 \in X$ be a limit point of E. Then $f(x_0)$ is a limit point of $f(E)$.

Solution. False.

Let the metric spaces be $X = Y = \mathbb{R}$, and let $f \colon X \to Y$ be $f(x) = 0$ for all $x \in \mathbb{R}$. Take $E = \mathbb{R}$ and $x_0 = 0$. Then x_0 is a limit point of E. However, $f(E) = \{0\}$ is is finite and so has no limit points. In particular, $f(x_0)$ isn't a limit point of $f(E)$. □

(f) Let X and Y be metric spaces, and let $f \colon X \to Y$ be a continuous function. Let $E \subset X$, and let $x_0 \in \overline{E}$. Then $f(x_0) \in \overline{f(E)}$.

Solution 1. True.

Since $x_0 \in \overline{E}$, there is a sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ in E such that $\lim_{n \to \infty} a_n = x_0$. Since f is continuous, $(f(a_n))_{n \in \mathbb{Z}_{>0}}$ is a sequence in $f(E)$ such that $\lim_{n \to \infty} f(a_n) = f(x_0)$. Therefore $f(x_0) \in \overline{f(E)}$. □

Solution 2. True.

We divide the proof in two cases: $x_0 \in E$ and x_0 is a limit point of E. If $x_0 \in E$ then $f(x_0) \in f(E) \subset \overline{f(E)}$. So suppose x_0 is a limit point of E. For each $n \in \mathbb{Z}_{>0}$ choose $a_n \in [N_\varepsilon(x_0) \setminus \{x_0\}] \cap E$. Then $\lim_{n \to \infty} a_n = x_0$. Since f is continuous, $(f(a_n))_{n \in \mathbb{Z}_{>0}}$ is a sequence in $f(E)$ such that $\lim_{n \to \infty} f(a_n) = f(x_0)$. Therefore $f(x_0) \in \overline{f(E)}$. □

Solution 3. True.

We prove that for every $\varepsilon > 0$ there is $y \in f(E)$ such that $d_Y(y, f(x_0)) < \varepsilon$. This certainly implies that $f(x_0) \in \overline{f(E)}$.

So let $\varepsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ such that whenever $x \in X$ satisfies $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$. Since $x_0 \in \overline{E}$, there is $x \in E$ such that $d_X(x, x_0) < \delta$. So $y = f(x) \in f(E)$ and satisfies $d_Y(y, f(x_0)) < \varepsilon$. □

2. (30 points) Let X be a metric space, and let K and L be compact subsets of X. Prove that $K \cup L$ is compact.

Solution. Let $\mathcal{U}$ be an open cover of $K \cup L$. Then $\mathcal{U}$ is in particular an open cover of K, so there is a finite subcollection $\mathcal{V} \subset \mathcal{U}$ which covers K. Similarly, $\mathcal{U}$ is an open cover of L, so there is a finite subcollection $\mathcal{W} \subset \mathcal{U}$ which covers K. Then $\mathcal{V} \cup \mathcal{W}$ is a finite subcollection of $\mathcal{U}$ which covers $K \cup L$. We have shown that every open cover of $K \cup L$ has a finite subcover, so $K \cup L$ is compact. □

3. (30 points) Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x+1 & x \neq 0 \\ 2 & x = 0. \end{cases}$$

Prove directly from the definition, with full details, that f is not continuous at 0, but that f is continuous at every other point of $\mathbb{R}$.

Solution. By definition, a function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in \mathbb{R}$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$. Therefore f is not continuous at x_0 if and only if there is $\varepsilon > 0$ such that for every $\delta > 0$ there is $x \in \mathbb{R}$ with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$.

We prove that f is not continuous at $x_0 = 0$. Take $\varepsilon = \frac{1}{2}$. Let $\delta > 0$. Set $x = -\frac{1}{2}\delta$. Then certainly $|x - x_0| = \frac{1}{2}\delta < \delta$. Also,

$$\begin{aligned} |f(x) - f(x_0)| &= \left|f\left(-\tfrac{1}{2}\delta\right) - f(0)\right| = \left|2\left(-\tfrac{1}{2}\delta\right) + 1 - 2\right| \\ &= |-\delta - 1| = \delta + 1 > \tfrac{1}{2} = \varepsilon. \end{aligned}$$

So f is not continuous at $x_0 = 0$.

Now let $x_0 \in \mathbb{R} \setminus \{0\}$; we prove that f is continuous at x_0. Let $\varepsilon > 0$. Set $\delta = \min\left(\frac{1}{2}\varepsilon,\ |x_0|\right)$. Then $\delta > 0$. Let $x \in \mathbb{R}$ satisfy $|x - x_0| < \delta$. In particular, we have $|x - x_0| < |x_0|$, so $x \neq 0$, whence $f(x) = 2x + 1$. Therefore

$$|f(x) - f(x_0)| = |2x + 1 - (2x_0 + 1)| = 2|x - x_0| < 2\left(\tfrac{1}{2}\varepsilon\right) = \varepsilon.$$

This shows that f is continuous at x_0. □

4. (20 points) Prove that there exists $x \in \mathbb{R}$ such that $x^{19} + 7x^3 + 11 = 0$.

Solution. Define $f(x) = x^{19} + 7x^3 + 11$ for $x \in \mathbb{R}$. Then f is a polynomial function, hence continuous. Also $f(0) = 11 > 0$ and

$$f(-2) = -2^{19} - 7 \cdot 2^3 + 11 < -7 \cdot 2^3 + 11 = -45 < 0.$$

Therefore the Intermediate Value Theorem provides $x \in (0, 2)$ such that $f(x) = 0$. □

A complete solution must explicitly exhibit (with proof) numbers $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$, or else provide a proof that such numbers exist (perhaps by showing that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$).

5. (30 points) Determine, with proof, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}.$$

Solution 1. Write the series as $\sum_{n=0}^{\infty} a_n z^n$ with

$$a_n = \begin{cases} \frac{1}{n} & \text{there is } k \in \mathbb{Z}_{\geq 0} \text{ such that } n = 2^k \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\limsup_{n\to\infty} \sqrt[n]{a_n} = 1$. To see this, first observe that $\sqrt[n]{a_n} \leq 1$ for all n, so that certainly $\limsup_{n\to\infty} \sqrt[n]{a_n} \leq 1$. On the other hand, the subsequence $\left(\sqrt[2^k]{a_{2^k}} \right)$ has limit (steps justified afterwards)

$$\lim_{k\to\infty} \sqrt[2^k]{a_{2^k}} = \lim_{k\to\infty} \left(\frac{1}{2^k} \right)^{1/2^k} = \lim_{n\to\infty} \left(\frac{1}{n} \right)^{1/n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = 1.$$

The second step is justified by observing that $\left((1/2^k)^{1/2^k} \right)_{n\in\mathbb{Z}_{\geq 0}}$ is a subsequence of $\left((\frac{1}{n})^{1/n} \right)_{n\in\mathbb{Z}_{>0}}$ and the second of these converges. The fourth step is from a theorem in Rudin's book. Since $\left(\sqrt[n]{a_n} \right)_{n\in\mathbb{Z}_{\geq 0}}$ has a subsequence converging to 1, it follows that $\limsup_{n\to\infty} \sqrt[n]{a_n} \geq 1$.

The radius of convergence is now

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}} = 1.$$

This completes the solution. □

Solution 2. We apply the ratio test to the given series. Write it as $\sum_{n=0}^{\infty} b_n(z)$ with

$$b_n(z) = \frac{z^{2^n}}{2^n}$$

for $n \in \mathbb{Z}_{\geq 0}$. Clearly the series converges when $z = 0$. For $z \neq 0$, we have

$$\lim_{n\to\infty} \frac{|b_{n+1}(z)|}{|b_n(z)|} = \lim_{n\to\infty} \left| \frac{z^{2^{n+1}}}{2^{n+1}} \right| \cdot \left| \frac{2^n}{z^{2^n}} \right| = \lim_{n\to\infty} \frac{|z|^{2^n}}{2}.$$

If $|z| > 1$ then $\lim_{n\to\infty} \frac{1}{2}|z|^n = \infty$. Therefore the original series diverges. If $|z| < 1$ then $\lim_{n\to\infty} \frac{1}{2}|z|^n = 0$. Therefore the original series converges when $0 < |z| < 1$. It follows immediately that the radius of convergence is 1. □

6. (30 points) Let $E \subset \mathbb{R}$ be an open interval, and let $f \colon E \to \mathbb{R}$ be differentiable. Suppose that f' is bounded on E. Prove that f is uniformly continuous.

Solution. By hypothesis, there is M such that $|f'(x)| \leq M$ for all $x \in E$. Let $\varepsilon > 0$. Set $\delta = M^{-1}\varepsilon$. Suppose $x, y \in E$ satisfy $|x - y| < \delta$. We claim that $|f(x) - f(y)| < \varepsilon$. To prove the claim, first assume $x < y$. By the Mean Value Theorem, there is $c \in (x, y)$ such that $f(x) - f(y) = f'(c)(x - y)$. Then

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| < M\delta = \varepsilon.$$

If instead $x > y$, exchange x and y in the argument just given. If $x = y$, the desired inequality is trivial. This completes the proof of the claim. □

Extra Credit. (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

Prove or disprove the following statement:

"Let $a, b \in \mathbb{R}$ satisfy $a < b$. Let $f,\ g\colon (a, b) \to \mathbb{R}$ be differentiable functions such that

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0,$$

and such that

$$\lim_{x \to 0} \frac{f(x)}{g(x)}$$

exists. Then the equation

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lim_{x \to a^+} \frac{f(x)}{g(x)}$$

holds."

(For the possible construction of counterexamples, you may use the standard properties of the trigonometric and exponential functions and their inverses from elementary calculus.)

Solution. The statement is false. Take $a = 0$ and $b = 1$, and define functions $f,\ g\colon (0, 1) \to \mathbb{R}$ by

$$f(x) = x^2 \sin\left(\frac{1}{x^2}\right) \qquad \text{and} \qquad g(x) = x$$

for $x \in (0, 1)$.

We have $\lim_{x \to 0^+} f(x) = 0$ because

$$|f(x)| = x^2 \left|\sin\left(\frac{1}{x^2}\right)\right| \leq x^2$$

for $x \neq 0$, and $\lim_{x \to 0^+} x^2 = 0$.

That $\lim_{x \to 0^+} g(x) = 0$ is trivial.

We have

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = 0$$

because

$$\left|\frac{f(x)}{g(x)}\right| = |x| \left|\sin\left(\frac{1}{x^2}\right)\right| \leq |x|$$

for $x \neq 0$, and $\lim_{x \to 0^+} |x| = 0$. In particular, this limit exists.

We claim that

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

does not exist. We first calculate

$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \left(\frac{2}{x}\right) \cos\left(\frac{1}{x^2}\right)$$

for $x \neq 0$, and $g'(x) = 1$ for all x. For $n \in \mathbb{Z}_{>0}$, set

$$x_n = \frac{1}{\sqrt{2\pi n}}.$$

Then $x_n > 0$ for all n, $\lim_{n \to \infty} x_n = 0$, and, using $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$,

$$\lim_{n \to \infty} 2x_n \sin\left(\frac{1}{x_n^2}\right) = 0.$$

However,

$$\lim_{n\to\infty} \left(\frac{2}{x_n}\right)\cos\left(\frac{1}{x_n^2}\right) = \lim_{n\to\infty} 2\sqrt{2\pi n}\cos(2\pi n) = \infty.$$

So $\lim_{n\to\infty} f'(x_n) = \infty$. Similarly, with

$$y_n = \frac{1}{\sqrt{(2n+1)\pi}}$$

for $n \in \mathbb{Z}_{>0}$, we get $\lim_{n\to\infty} f'(y_n) = -\infty$. So $\lim_{x\to 0} f'(x)$ does not exist. Since $g'(x) = 1$ for all x, the claim follows. $\square$

The solution given is a slight overkill. We only need to show that $\frac{f'(x)}{g'(x)} \not\to 0$ as $x \to 0^+$, not that the limit doesn't exist. For this purpose, it suffices to use the sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$.