MATH 413[513] (PHILLIPS) MIDTERM: 2 NOV. 2001

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Write your name and your student ID on your paper.

Total: 120 points; time: 120 minutes.

1. (30 points) Let X be a metric space, let $Y \subset X$, and let $E \subset Y$. Prove that E is closed as a subset of Y if and only if there is a closed subset F of X such that $E = Y \cap F$.

2. (30 points) A sequence in $(x_n)_{n \in \mathbb{Z}_{>0}}$ a metric space X is *bounded* if there are $r \in (0, \infty)$ and $x \in X$ such that $x_n \in N_r(x)$ for all $n \in \mathbb{Z}_{>0}$.
Prove directly from the definition that every Cauchy sequence is bounded.

3. (30 points) Let

$$K = \left\{ \tfrac{1}{n} \colon n \in \mathbb{Z}, \ n \neq 0 \right\} \cup \{0\} \subset \mathbb{R}.$$

Prove directly from the definition that K is a compact subset of $\mathbb{R}$. (No credit will be given for using the theorem about closed bounded subsets of $\mathbb{R}^n$.)

4. (15 points) Does the the series

$$\sum_{n=2}^{\infty} \frac{1}{(n + (-1)^n)^2}$$

converge? Cite any theorems you use and verify that the hypotheses are met to justify your answer.

5. (15 points) Determine, with proof, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} \cdot z^{2n}.$$

(Be careful!)

Extra Credit. (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 75% on the rest of the exam.)

Let X and Y be metric spaces, with metrics d_X and d_Y. Define a metric d on $X \times Y$ by

$$d\big((x_1, y_1), \, (x_2, y_2)\big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

(a) Prove that d is in fact a metric.

(b) Let $K \subset X$ be compact, let $y_0 \in Y$, and let $W \subset X \times Y$ be an open set such that $K \times \{y_0\} \subset W$. Prove that there are an open set $U \subset X$ with $K \subset U$ and an open set $V \subset Y$ with $y_0 \in Y$ such that $U \times V \subset W$.