MATH 413 [PHILLIPS] MIDTERM: 2 NOV. 2001

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Write your name and your student ID on your paper.

Total: 120 points; time: 120 minutes.

1. (30 points) Let \( X \) be a metric space, let \( Y \subset X \), and let \( E \subset Y \). Prove that \( E \) is closed as a subset of \( Y \) if and only if there is a closed subset \( F \) of \( X \) such that \( E = Y \cap F \).

2. (30 points) A sequence in \((x_n)_{n \in \mathbb{Z} > 0}\) a metric space \( X \) is bounded if there are \( r \in (0, \infty) \) and \( x \in X \) such that \( x_n \in N_r(x) \) for all \( n \in \mathbb{Z} > 0 \).
Prove directly from the definition that every Cauchy sequence is bounded.

3. (30 points) Let \( K = \left\{ \frac{1}{n} : n \in \mathbb{Z}, \ n \neq 0 \right\} \cup \{0\} \subset \mathbb{R} \).
Prove directly from the definition that \( K \) is a compact subset of \( \mathbb{R} \). (No credit will be given for using the theorem about closed bounded subsets of \( \mathbb{R}^n \).)

4. (15 points) Does the series
\[
\sum_{n=2}^{\infty} \frac{1}{(n + (-1)^n)^2}
\]
converge? Cite any theorems you use and verify that the hypotheses are met to justify your answer.

5. (15 points) Determine, with proof, the radius of convergence of the power series
\[
\sum_{n=0}^{\infty} \frac{1}{3^n} \cdot z^{2n}.
\]
(Be careful!)
Extra Credit. (Don’t do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 75% on the rest of the exam.)

Let \( X \) and \( Y \) be metric spaces, with metrics \( d_X \) and \( d_Y \). Define a metric \( d \) on \( X \times Y \) by
\[
d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.
\]
(a) Prove that \( d \) is in fact a metric.

(b) Let \( K \subset X \) be compact, let \( y_0 \in Y \), and let \( W \subset X \times Y \) be an open set such that \( K \times \{y_0\} \subset W \). Prove that there are an open set \( U \subset X \) with \( K \subset U \) and an open set \( V \subset Y \) with \( y_0 \in Y \) such that \( U \times V \subset W \).