MATH 413[513] (PHILLIPS) SOLUTIONS TO MIDTERM

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Write your name and your student ID on your paper.

Total: 120 points; time: 120 minutes.

1. (30 points) Let X be a metric space, let $Y \subset X$, and let $E \subset Y$. Prove that E is closed as a subset of Y if and only if there is a closed subset F of X such that $E = Y \cap F$.

Solution. Assume that E is closed as a subset of Y. Then $Y \setminus E$ is open as a subset of Y. Therefore, by Theorem 2.30 of Rudin, there is an open subset V of X such that $Y \setminus E = Y \cap V$. Set $F = X \setminus V$. Then F is a closed subset of X and

$$Y \cap F = Y \setminus (Y \cap V) = Y \setminus (Y \setminus E) = E.$$

Conversely, assume that there is a closed subset F of X such that $E = Y \cap F$. Then $X \setminus F$ is an open subset of X. Theorem 2.30 of Rudin implies that $Y \cap (X \setminus F)$ is an open subset of Y. Since $E = Y \cap F = Y \setminus [Y \cap (X \setminus F)]$, it follows that E is closed as a subset of Y. $\square$

Remark: In this problem, it is inappropriate to use the notation E^{c} for the complement of E, because the proof uses complements with respect to both X and Y.

Alternate solution. Assume that E is closed as a subset of Y. Let F be the closure of E in X. Then F is a closed subset of X.

We show that $E = Y \cap F$. Clearly $E \subset Y \cap F$. For the reverse inclusion, let $x \in Y \cap F$; we show that $x \in E$. Since x is in the closure of E in X, it follows that $x \in E$ or x is a limit point of E in X. If $x \in E$, we are done. So suppose that x is a limit point of E in X. First, $x \in Y$ because $x \in Y \cap F$. Second, we claim that x is a limit point of E in Y. To prove the claim, let $\varepsilon > 0$. Let $N_\varepsilon(x)$ be the open ε-ball about x in X. Then there is a point $y \in E \cap N_\varepsilon(x)$ such that $y \neq x$. The open ε-ball about x in Y is $Y \cap N_\varepsilon(x)$. Since $y \in E \subset Y$, it follows that y is a point in $E \cap [Y \cap N_\varepsilon(x)]$ such that $y \neq x$. This proves the claim. Since E is closed in Y, it follows that $x \in E$.

We have shown that $E = Y \cap F$, so the "only if" part of the proof is done.

Conversely, assume that there is a closed subset F of X such that $E = Y \cap F$. We show that E is closed in Y, by proving that every limit point of E in Y is actually in E. So let $x \in Y$ and suppose that x is a limit point of E in Y. We claim that x is a limit point of F in X. To prove the claim, let $\varepsilon > 0$, again let $N_\varepsilon(x)$ be the open ε-ball about x in X, recall that the open ε-ball about x in Y is $Y \cap N_\varepsilon(x)$, and use the fact that x is a limit point of E in Y to choose $y \in E \cap [Y \cap N_\varepsilon(x)]$ such that $y \neq x$. Then in particular $y \in F \cap N_\varepsilon(x)$ and $y \neq x$. Since $\varepsilon > 0$ is arbitrary, the claim follows. Now F is closed in X, so $x \in F$. We already know $x \in Y$, so

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we have $x \in Y \cap F = E$. Thus every limit point of E in Y is already in E, and we have shown that E is closed in Y. $\square$

2. (30 points) A sequence in $(x_n)_{n \in \mathbb{Z}_{>0}}$ a metric space X is *bounded* if there are $r \in (0, \infty)$ and $x \in X$ such that $x_n \in N_r(x)$ for all $n \in \mathbb{Z}_{>0}$.

Prove directly from the definition that every Cauchy sequence is bounded.

Solution. Let X be a metric space, and let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in X. Choose N such that if $m, n \geq N$ then $d(x_m, x_n) < 1$. Let

$$r = 1 + \max_{1 \leq k \leq N-1} d(x_N, x_k).$$

Clearly r is finite. We claim that $\{x_n : n \in \mathbb{Z}_{>0}\} \subset N_r(x_N)$.

We prove the claim. If $k < N$, then $d(x_k, x_N) < r$ by construction. If $k \geq N$, then $d(x_k, x_N) < 1 < r$ by the choice of N. The claim is proved. $\square$

3. (30 points) Let

$$K = \left\{ \tfrac{1}{n} : n \in \mathbb{Z},\ n \neq 0 \right\} \{0\} \cup \left\{ \tfrac{1}{n} : n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}.$$

Prove directly from the definition that K is a compact subset of $\mathbb{R}$. (No credit will be given for using the theorem about closed bounded subsets of $\mathbb{R}^n$.)

Solution. We verify that every open cover of K has a finite subcover.

Let $(U_i)_{i \in I}$ be an open cover of K. Choose $j \in I$ such that $0 \in U_j$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U_j$. Choose $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{N} < \varepsilon$. Then $|n| \geq N$ implies $\left|\frac{1}{n}\right| < \varepsilon$, so $\frac{1}{n} \in U_j$. For $n = 1, 2, \ldots, N-1$, choose $k(n) \in I$ such that $\frac{1}{n} \in U_{k(n)}$, and choose $l(n) \in I$ such that $-\frac{1}{n} \in U_{l(n)}$. Set

$$F = \left\{ j,\ k(1),\ k(2),\ \ldots,\ k(N-1),\ l(1),\ l(2),\ \ldots,\ l(N-1) \right\} \subset I.$$

Then $(U_i)_{i \in F}$ is a finite subfamily of $(U_i)_{i \in I}$ which covers K. $\square$

It is also correct to say that a set K is compact if and only if whenever $\mathcal{U}$ is a collection of open sets which covers K, then there is a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ which covers K. (The only difference in the formulation is that this version doesn't allow for repetition of the sets. Sometimes it is not convenient to allow repetition.)

Since people have used this formulation of the definition, but gotten the notation wrong, we give the proof using this formulation.

Alternate solution. We verify that every open cover of K has a finite subcover.

Let $\mathcal{U}$ be an open cover of K. Choose $V \in \mathcal{U}$ such that $0 \in V$. Choose $\varepsilon > 0$ such that $N_\varepsilon(0) \subset V$. Choose $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{N} < \varepsilon$. Then $|n| \geq N$ implies $\left|\frac{1}{n}\right| < \varepsilon$, so $\frac{1}{n} \in V$ ~~. For $n \in \mathbb{Z}_{>0}$~~ and $-\frac{1}{n} \in V$. For $n \in \mathbb{Z}$ with $1 \leq |n| < N$, choose $V_n \in \mathcal{U}$ such that $x_n \in V_n$. Then

$$\mathcal{V} = \{V,\ V_1,\ V_{-1},\ V_2,\ V_{-2},\ \ldots,\ V_{N-1},\ V_{-(N-1)}\}$$

is a finite subcollection of $\mathcal{U}$ which covers K. $\square$

4. (15 points) Does the the series

$$\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$$

converge? Cite any theorems you use and verify that the hypotheses are met to justify your answer.

Solution 1. We use the comparison test.

Set $c_n = (n+(-1)^n)^2$. For $n \geq 2$, we have $n+(-1)^n \geq n-1 \geq 0$, so

$$c_n = (n+(-1)^n)^2 \geq (n-1)^2.$$

Therefore

$$0 < \frac{1}{(n+(-1)^n)^2} \leq \frac{1}{(n-1)^2}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$ converges, because it is equal to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is known to converge. Therefore

$$\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$$

converges by the comparison test. □

Solution 2. We use the comparison test.

Set $c_n = (n+(-1)^n)^2$. For $n \geq 2$, we have $n+(-1)^n \geq n-1 \geq \frac{1}{2}n > 0$, so

$$c_n = (n+(-1)^n)^2 \geq \left(\tfrac{1}{2}n\right)^2 = \tfrac{1}{4}n^2.$$

Therefore

$$0 < \frac{1}{(n+(-1)^n)^2} \leq \frac{4}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges. Therefore

$$\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$$

converges by the comparison test. □

Solution 3. The series is

$$\frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{4^2} + \frac{1}{7^2} + \frac{1}{6^2} + \cdots,$$

which is a rearrangement of $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

To make this precise, define $\sigma \colon \mathbb{Z}_{>0} \setminus \{1\} \to \mathbb{Z}_{>0} \setminus \{1\}$ by

$$\sigma(n) = \begin{cases} n+1 & n \text{ even} \\ n-1 & n \text{ odd}. \end{cases}$$

Then σ is bijective. With

$$a_n = \frac{1}{(n+(-1)^n)^2} \qquad \text{and} \qquad b_n = \frac{1}{n^2}$$

for $n \in \{2, 3, \ldots\}$, we have $a_n = b_{\sigma(n)}$ for all $n \geq 2$. Moreover, $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges absolutely, so we may apply the theorem on rearrangements of absolutely convergent series to get

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} a_{\sigma(n)} = \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2},$$

with absolute convergence everywhere. In particular,

$$\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$$

converges. □

Remark: With the particular σ used in Solution 3, one can prove that

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} a_{\sigma(n)}$$

even without absolute convergence. (If you use this fact, though, you must prove it.)

5. (15 points) Determine, with proof, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} \cdot z^{2n}.$$

(Be careful!)

Solution 1. We write the series as

$$\sum_{n=0}^{\infty} a_n z^n$$

with

$$a_n = \begin{cases} 3^{-n/2} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Then

$$\sqrt[n]{|a_n|} = \begin{cases} 1/\sqrt{3} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

This sequence clearly has two subsequential limits, namely $1/\sqrt{3}$ and 0. So

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} = \frac{1}{\sqrt{3}},$$

from which it follows that the power series has radius of convergence equal to $\sqrt{3}$. □

Solution 2. We use the ratio test to show that the series converges for $|z| < \sqrt{3}$ and diverges for $|z| > \sqrt{3}$.

For $z = 0$, the series converges for trivial reasons.

For $z \neq 0$, we observe that the ratios of the absolute values of the terms are

$$\left| \frac{1}{3^{n+1}} \cdot z^{2(n+1)} \right| \cdot \left| \frac{1}{3^n} \cdot z^{2n} \right|^{-1} = \frac{|z|^2}{3}.$$

We therefore clearly have

$$\lim_{n\to\infty} \left| \frac{1}{3^{n+1}} \cdot z^{2(n+1)} \right| \cdot \left| \frac{1}{3^n} \cdot z^{2n} \right|^{-1} = \frac{|z|^2}{3}.$$

If $|z| < \sqrt{3}$ then $\frac{1}{3}|z|^2 < 1$, so the series converges by the ratio test.

If $|z| > \sqrt{3}$ then $\frac{1}{3}|z|^2 > 1$, so the series diverges by the ratio test.

We have shown that the series converges for $|z| < \sqrt{3}$ and diverges for $|z| > \sqrt{3}$, from which it is immediate that the radius of convergence is $\sqrt{3}$. □

Solution 3. We first consider the power series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} \cdot w^n.$$

(Note that the exponent is different.) Write this series as $\sum_{n=0}^{\infty} a_n w^n$ with $a_n = 3^{-n}$ for all n. Then it has radius of convergence

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|3^{-n}|}} = \frac{1}{\limsup_{n\to\infty} \frac{1}{3}} = 3.$$

Thus, $\sum_{n=0}^{\infty} a_n w^n$ converges for $|w| < 3$ and diverges for $|w| > 3$.

The series we are interested in is $\sum_{n=0}^{\infty} a_n (z^2)^n$, which therefore converges when $|z^2| < 3$ and diverges when $|z^2| > 3$. That is, it converges when $|z| < \sqrt{3}$ and diverges when $|z| > \sqrt{3}$. It is now immediate that the radius of convergence is $\sqrt{3}$. □

Extra Credit. (Don't do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 75% on the rest of the exam.)

Let X and Y be metric spaces, with metrics d_X and d_Y. Define a metric d on $X \times Y$ by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

(a) Prove that d is in fact a metric.

Solution (sketch). For $z_1, z_2 \in X \times Y$, the proofs that $d(z_1, z_2) \geq 0$, that $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$, and that $d(z_1, z_2) = d(z_2, z_1)$, are all routine, and are omitted.

For the triangle inequality, we use the following lemma.

Lemma. Let $n \in \mathbb{Z}_{>0}$ and let

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in [0, \infty)$$

satisfy $a_k \leq b_k + c_k$ for $1 \leq k \leq n$. Then

$$\sqrt{\textstyle\sum_{k=1}^n a_k^2} \leq \sqrt{\textstyle\sum_{k=1}^n b_k^2} + \sqrt{\textstyle\sum_{k=1}^n c_k^2}.$$

Proof. Write $a_k = x_k + y_k$ with $0 \leq x_k \leq b_k$ and $0 \leq y_k \leq c_k$ for $1 \leq k \leq n$. Then

$$a = (a_1, a_2, \ldots, a_n),\ x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n.$$

and $a = x + y$. The triangle inequality for the usual norm on $\mathbb{R}^n$ therefore gives $\|a\| \leq \|x\| + \|y\|$. Written in coordinates, this says that

$$\sqrt{\sum_{k=1}^n a_k^2} \leq \sqrt{\sum_{k=1}^n x_k^2} + \sqrt{\sum_{k=1}^n y_k^2}.$$

The desired result now follows from the inequalities

$$\sqrt{\sum_{k=1}^n x_k^2} \leq \sqrt{\sum_{k=1}^n b_k^2} \qquad \text{and} \qquad \sqrt{\sum_{k=1}^n y_k^2} \leq \sqrt{\sum_{k=1}^n c_k^2}.$$

This completes the proof. $\square$

With this lemma in hand, let $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3) \in X \times Y$. Then

$$d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3) \qquad \text{and} \qquad d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3)$$

by the triangle inequality in X and the triangle inequality in Y. To get

$$d\big((x_1, y_1),\ (x_3, y_3)\big) \leq d\big((x_1, y_1),\ (x_2, y_2)\big) + d\big((x_2, y_2),\ (x_3, y_3)\big),$$

apply the lemma with $n = 2$ and with

$$a_1 = d_X(x_1, x_3), \quad b_1 = d_X(x_1, x_2), \qquad \text{and} \qquad c_1 = d_X(x_2, x_3)$$

and

$$a_2 = d_Y(y_1, y_3), \quad b_2 = d_Y(y_1, y_2), \qquad \text{and} \qquad c_2 = d_Y(y_2, y_3).$$

This completes the solution. $\square$

Alternate solution (sketch). As in the first solution, all properties except the triangle inequality are easy. For the triangle inequality, we use the following lemma (the case $n = 2$ of the lemma in the previous solution).

Lemma. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in [0, \infty)$ satisfy

$$a_1 \leq b_1 + c_1 \qquad \text{and} \qquad a_2 \leq b_2 + c_2.$$

Then

$$\sqrt{a_1^2 + a_2^2} \leq \sqrt{b_1^2 + b_2^2} + \sqrt{c_1^2 + c_2^2}.$$

Proof. We use the following sequence of inequalities, each of which follows from the previous one:

$$(b_1 c_2 - b_2 c_1)^2 \geq 0.$$

$$2 b_1 c_1 b_2 c_2 \leq b_1^2 c_2^2 + b_2^2 c_1^2.$$

$$b_1^2 c_1^2 + 2 b_1 c_1 b_2 c_2 + b_2^2 c_2^2 \leq b_1^2 c_1^2 + b_1^2 c_2^2 + b_2^2 c_1^2 + b_2^2 c_2^2.$$

$$(b_1 c_1 + b_2 c_2)^2 \leq (b_1^2 + b_2^2)(c_1^2 + c_2^2).$$

$$2 b_1 c_1 + 2 b_2 c_2 \leq 2 \left(\sqrt{b_1^2 + b_2^2} \right) \left(\sqrt{c_1^2 + c_2^2} \right).$$

$$(b_1 + c_1)^2 + (b_2 + c_2)^2 \leq b_1^2 + b_2^2 + c_1^2 + c_2^2 + 2 \left(\sqrt{b_1^2 + b_2^2} \right) \left(\sqrt{c_1^2 + c_2^2} \right).$$

$$a_1^2 + a_2^2 \leq \left(\sqrt{b_1^2 + b_2^2} + \sqrt{c_1^2 + c_2^2} \right)^2.$$

$$\sqrt{a_1^2 + a_2^2} \leq \sqrt{b_1^2 + b_2^2} + \sqrt{c_1^2 + c_2^2}.$$

This completes the proof. $\square$

With this lemma in hand, the proof is completed as in the first solution. $\square$

Remark: The steps in the proof of the lemma were, of course, found by working backwards.

(b) Let $K \subset X$ be compact, let $y_0 \in Y$, and let $W \subset X \times Y$ be an open set such that $K \times \{y_0\} \subset W$. Prove that there are an open set $U \subset X$ with $K \subset U$ and an open set $V \subset Y$ with $y_0 \in Y$ such that $U \times V \subset W$.

Solution. For convenience, we let $N_\varepsilon(x, y)$ denote the open ε-ball in $X \times Y$ about $(x, y) \in X \times Y$, we let $N_\varepsilon^X(x)$ denote the open ε-ball in X about $x \in X$, and we let $N_\varepsilon^Y(y)$ denote the open ε-ball in Y about $y \in Y$. We further observe that for any $\varepsilon > 0$, any $x \in X$, and any $y \in Y$, we have

$$N_{\varepsilon/2}^X(x) \times N_{\varepsilon/2}^Y(y) \subset N_\varepsilon(x, y)$$

(In fact, this works for $\varepsilon/\sqrt{2}$ in place of $\varepsilon/2$, but $\varepsilon/2$ is good enough for our purposes.)

For each $x \in K$, use the fact that W is open to choose $\varepsilon(x) > 0$ such that $N_{\varepsilon(x)}(x, y_0) \subset W$. The sets $N_{\varepsilon(x)/2}^X(x)$, for $x \in K$, form an open cover of K. Since K is compact, there are $x_1, x_2, \ldots, x_n \in K$ such that the sets

$$N_{\varepsilon(x_1)/2}^X(x_1),\ N_{\varepsilon(x_2)/2}^X(x_2),\ \ldots,\ N_{\varepsilon(x_n)/2}^X(x_n)$$

cover K. Let

$$\varepsilon = \min\left(\tfrac{1}{2}\varepsilon(x_1),\ \tfrac{1}{2}\varepsilon(x_2),\ \ldots,\ \tfrac{1}{2}\varepsilon(x_n)\right).$$

Then $\varepsilon > 0$. Define

$$U = \bigcup_{k=1}^{n} N_{\varepsilon(x_k)/2}^X(x_k) \qquad \text{and} \qquad V = N_\varepsilon^Y(y_0).$$

Then U is an open subset of X which contains K and V is an open subset of Y which contains y_0.

It remains only to prove that $U \times V \subset W$. So let $x \in U$ and $y \in V$. Then there is k such that $x \in N_{\varepsilon(x_k)/2}^X(x_k)$. Since $\varepsilon \leq \frac{1}{2}\varepsilon(x_k)$, we have $y \in N_{\varepsilon(x_k)/2}^Y(y_0)$. Therefore

$$(x, y) \in N_{\varepsilon(x_k)/2}^X(x_k) \times N_{\varepsilon(x_k)/2}^Y(y_0) \subset N_{\varepsilon(x_k)}(x_k, y_0) \subset U.$$

This completes the solution. $\square$