Generally, a “solution” is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A “solution (sketch)” is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.2. Prove that the set of algebraic numbers is countable.

Solution. For each fixed integer \( n \geq 0 \), the set \( P_n \) of all polynomials with integer coefficients and degree at most \( n \) is countable, since it has the same cardinality as the set
\[
\{(a_0, \ldots, a_n) : a_0, a_1, \ldots, a_n \in \mathbb{Z} \geq 0\} = (\mathbb{Z} \geq 0)^{n+1}.
\]
The set of all polynomials with integer coefficients is \( \bigcup_{n=0}^{\infty} P_n \), which is a countable union of countable sets and so countable. Each nonzero polynomial has only finitely many roots (at most \( n \) for degree \( n \)), so the set of all possible roots of all nonzero polynomials with integer coefficients is a countable union of finite sets, hence countable. □

Problem 2.3. Prove that there exist real numbers which are not algebraic.

Solution (sketch). This follows from Problem 2.2, since \( \mathbb{R} \) is not countable. □

Problem 2.4. Is \( \mathbb{R} \setminus \mathbb{Q} \) countable?

Solution (sketch). No. The set \( \mathbb{Q} \) is countable and \( \mathbb{R} \) is not countable. □

Problem 2.5. Construct a bounded subset of \( \mathbb{R} \) with exactly 3 limit points.

Solution (sketch). For example, the limit points of the set
\[
E = \{1/n : n \in \mathbb{Z} > 0\} \cup \{1 + 1/n : n \in \mathbb{Z} > 0\} \cup \{2 + 1/n : n \in \mathbb{Z} > 0\}
\]
are 0, 1, and 2. □

A correct solution includes a proof that 0, 1, and 2 are limit points of \( E \), and also a proof that \( E \) has no other limit points.

Problem 2.6. For a metric space \( X \) and a subset \( E \subset X \), let \( E' \) denote the set of limit points of \( E \).

1. Prove that \( E' \) is closed.
2. Prove that \( (E')' = E' \).
3. Is \( (E')' \) always equal to \( E' \)?

Solution to (1). We claim that \( (E')' \subset E' \). To prove the claim, let \( x \in (E')' \) and let \( \varepsilon > 0 \). By the definition of \( (E')' \), there is \( y \in E' \cap (N_{\varepsilon}(x) \setminus \{x\}) \). Define
\[
\delta = \min\{d(x, y), \varepsilon - d(x, y)\}.
\]

Date: 8 October 2018.
Then $\delta > 0$. By the definition of $E'$, there is $z \in E \cap (N_\delta(y) \setminus \{y\})$. Then
\[ d(z, y) < \delta \leq d(x, y), \]
so $z \neq x$. Also,
\[ d(z, x) \leq d(z, y) + d(y, x) < \delta + d(y, x) \leq (\varepsilon - d(x, y)) + d(x, y) = \varepsilon, \]
so $z \in N_\varepsilon(x)$. This completes the proof of the claim.

The claim implies that all limit points of $E'$ are in $E'$, so $E'$ is closed. \square

Here is a different way to prove that $(E')' \subset E'$.

Alternate solution to (1) (sketch). Let $x \in (E')'$ and let $\varepsilon > 0$. By definition, there is a point $y \in E' \cap (N_{\varepsilon/2}(x) \setminus \{x\})$. By Theorem 2.20 of Rudin, there are infinitely many points in $E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$. In particular there is a point $z \in E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$ with $z \neq x$. Now $z \in E \cap (N_\varepsilon(x) \setminus \{x\})$

Finish as in the first solution. \square

Solution to (2). We first claim that if $A$ and $B$ are any subsets of $X$, then $(A \cup B)' \subset A' \cup B'$. The fastest way to prove the claim is to assume that $x \in (A \cup B)'$ but $x \notin A'$, and to show that $x \notin B'$. Accordingly, let $x \in (A \cup B)' \setminus A'$. Since $x \notin A'$, there is $\varepsilon_0 > 0$ such that $N_{\varepsilon_0}(x) \cap A \subset \{x\}$.

Now let $\varepsilon > 0$; we show that $N_\varepsilon(x) \cap B$ contains at least one point different from $x$. To do so, set $r = \min(\varepsilon, \varepsilon_0) > 0$. Because $x \in (A \cup B)'$, there is $y \in N_r(x) \cap (A \cup B)$ with $y \neq x$. Then $y \notin A$ because $r \leq \varepsilon_0$. So necessarily $y \in B$. We have shown that $N_\varepsilon(x) \cap B$ contains at least one point, namely $y$, which is different from $x$.

The previous paragraph shows that $x \in B'$, and completes the proof of the claim.

We also claim that if $A \subset B \subset X$ then $A' \subset B'$. The proof of this claim is obvious, since if $N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$, then certainly $N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \emptyset$.

We now prove that $(E)' \subset E'$. Using the first claim at the second step and $(E)' \subset E'$ (proved in the proof of Part (2)) at the third step, we have
\[ (E)' = (E \cup E)' \subset E' \cup (E')' \subset E' \cup E' = E', \]
as desired. The inclusion $E' \subset (E)'$ follows from the second claim. \square

Alternate solution to (2) (sketch). An alternate proof that $(E)' \subset E'$ can be obtained by slightly modifying either of the proofs above that $(E)' \subset E'$. \square

Solution to (3). The answer is no. Take
\[ E = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{Z}_{>0} \right\}. \]
Then $E' = \{0\}$ and $(E)' = \emptyset$. (Of course, you must prove these facts.) \square

Problem 2.8. If $E \subset \mathbb{R}^2$ is open, is every point of $E$ a limit point of $E$? What if $E$ is closed instead of open?

Solution (sketch). Every point of an open set $E \subset \mathbb{R}^2$ is a limit point of $E$. Indeed, if $x \in E$, then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$, and it is easy to show that $x$ is a limit point of $N_\varepsilon(x)$.

Not every point of a closed set need be a limit point. Take $E = \{(0,0)\}$, which has no limit points. \square
Warning: in a general metric space, it need not be true that every point in an open set is a limit point of that set.

**Problem 2.9.** For a metric space $X$ and a subset $E \subset X$, let $E^\circ$ denote the set of interior points of a set $E$, that is, the interior of $E$.

1. Prove that $E^\circ$ is open.
2. Prove that $E$ is open if and only if $E^\circ = E$.
3. If $G$ is open and $G \subset E$, prove that $G \subset E^\circ$.
4. Prove that $X \setminus E^\circ = X \setminus E$.
5. Prove or disprove: $(E^\circ)^\circ = E^\circ$.
6. Prove or disprove: $\overline{E^\circ} = \overline{E}$.

**Solution to (1).** Let $x \in E^\circ$. Then there is $\epsilon > 0$ such that $N_\epsilon(x) \subset E$.

We claim that $N_\epsilon(x) \subset E^\circ$. To prove the claim, let $y \in N_\epsilon(x)$. Since $N_\epsilon(x)$ is open, there is $\delta > 0$ such that $N_\delta(y) \subset N_\epsilon(x)$. So $N_\delta(y) \subset E$. This shows that $y \in E^\circ$, proving the claim.

We have shown that for every $x \in E^\circ$ there is $\epsilon > 0$ such that $N_\epsilon(x) \subset E^\circ$. That is, $E^\circ$ is open. □

**Solution to (2).** If $E$ is open, then $E = E^\circ$ by the definition of $E^\circ$. If $E = E^\circ$, then $E$ is open by Part (a). □

**Solution to (3) (sketch).** We first claim that if $A \subset B \subset X$, then $A^\circ \subset B^\circ$. To prove the claim, let $x \in A^\circ$. By definition, there is $\epsilon > 0$ such that $N_\epsilon(x) \subset A$. Then also $N_\epsilon(x) \subset B$, so $x \in B^\circ$. The claim is proved.

Now let $G \subset X$ be an open set such that $G \subset E$. Using (2) at the first step and the claim at the second step, we have

$$G = G^\circ \subset E^\circ,$$

as desired. □

**Solution to (4) (sketch).** First show that $X \setminus E^\circ \subset X \setminus E$. If $x \notin E$, then clearly $x \in X \setminus E$. Otherwise, consider $x \in E \setminus E^\circ$. Rearranging the statement that $x$ fails to be an interior point of $E$, and noting that $x$ itself is not in $X \setminus E$, one gets exactly the statement that $x$ is a limit point of $X \setminus E$.

Now show that $X \setminus E \subset X \setminus E^\circ$. If $x \in X \setminus E$, then clearly $x \notin E^\circ$. If $x \notin X \setminus E$ but $x$ is a limit point of $X \setminus E$, then one simply rearranges the definition of a limit point to show that $x$ is not an interior point of $E$. □

**Solution to (5) (sketch).** This is false. Example: take $X = \mathbb{R}$ and $E = (0, 1) \cup (1, 2)$. We have $E^\circ = E$, $\overline{E} = [0, 2]$, and $(\overline{E})^\circ = (0, 2)$. □

**Alternate solution to (5) (sketch).** This is false. Example: take $X = \mathbb{R}$ and $E = \mathbb{Q}$. We have $E^\circ = \emptyset$, $\overline{E} = \mathbb{R}$, and $(\overline{E})^\circ = \mathbb{R}$. □

**Solution to (6) (sketch).** This is false. Example: take $X = \mathbb{R}$ and $E = (0, 1) \cup \{2\}$. Then $\overline{E} = [0, 1] \cup \{2\}$, $E^\circ = (0, 1)$, and $\overline{E^\circ} = [0, 1]$. □

**Alternate solution to (6) (sketch).** This is false. Example: take $X = \mathbb{R}$ and $E = \mathbb{Q}$. Then $\overline{E} = \mathbb{R}$, $E^\circ = \emptyset$, and $\overline{E^\circ} = \emptyset$. □

**Second alternate solution to (6) (sketch).** This is false. Example: take $X = \mathbb{R}$ and $E = \{0\}$. Then $\overline{E} = \{0\}$, $E^\circ = \emptyset$, and $\overline{E^\circ} = \emptyset$. □
Problem 2.11. Which of the following are metrics on $\mathbb{R}$?

1. $d_1(x, y) = (x - y)^2$ for $x, y \in \mathbb{R}$.
2. $d_2(x, y) = \sqrt{|x - y|}$ for $x, y \in \mathbb{R}$.
3. $d_3(x, y) = |x^2 - y^2|$ for $x, y \in \mathbb{R}$.
4. $d_4(x, y) = |x - 2y|$ for $x, y \in \mathbb{R}$.
5. $d_5(x, y) = \frac{|x - y|}{1 + |x - y|}$ for $x, y \in \mathbb{R}$.

Solution to (1) (sketch). No. The triangle inequality fails with $x = 0$, $y = 2$, and $z = 4$.

Solution to (2) (sketch). Yes. Some work is needed to check the triangle inequality.

Solution to (3) (sketch). No. We have $d_3(1, -1) = 0$.

Solution to (4) (sketch). No. We have $d_4(1, 1) \neq 0$.

Alternate solution to (4) (sketch). No. We have $d_4(1, 6) \neq d_4(6, 1)$.

Solution to (5). Yes.

It is obvious that $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and that $d(x, y) = 0$ if and only if $x = y$. It is also obvious that $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}$.

It remains to prove the triangle inequality.

We first claim that the function $t \mapsto \frac{t}{1 + t}$ is nondecreasing on $[0, \infty)$. To prove the claim, let $s, t \in \mathbb{R}$ satisfy $0 \leq s \leq t$. Then

$$\frac{t}{1 + t} - \frac{s}{1 + s} = \frac{t(1 + s) - s(1 + t)}{(1 + t)(1 + s)} = \frac{t - s}{(1 + t)(1 + s)} \geq 0.$$  

This proves the claim.

We next claim that if $a, b \in [0, \infty)$ then

$$\frac{a + b}{1 + a + b} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.$$  

To prove the claim, we calculate:

$$\frac{a + b}{1 + a + b} = \frac{a}{1 + a + b} + \frac{b}{1 + a + b} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.$$  

The claim is proved.

Now let $x, y, z \in \mathbb{R}$. Then, using $|x - z| \leq |x - y| + |y - z|$ and the first claim at the second step and using the second claim at the third step,

$$d_5(x, z) = \frac{|x - z|}{1 + |x - z|} \leq \frac{|x - y| + |y - z|}{1 + |x - y| + |y - z|} \leq \frac{|x - y|}{1 + |x - y|} + \frac{|y - z|}{1 + |y - z|} = d_5(x, y) + d_5(y, z).$$

This completes the solution.

The proof given is easier than what most people did the last time I assigned this problem.

The first claim can also be proved using elementary calculus. This method isn’t really legitimate because we haven’t done any calculus yet.