MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 2

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.2. Prove that the set of algebraic numbers is countable.

Solution. For each fixed integer $n \geq 0$, the set P_n of all polynomials with integer coefficients and degree at most n is countable, since it has the same cardinality as the set

$$\big\{ (a_0, \ldots, a_n) \colon a_0, a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0} \big\} = (\mathbb{Z}_{\geq 0})^{n+1}.$$

The set of all polynomials with integer coefficients is $\bigcup_{n=0}^{\infty} P_n$, which is a countable union of countable sets and so countable. Each nonzero polynomial has only finitely many roots (at most n for degree n), so the set of all possible roots of all nonzero polynomials with integer coefficients is a countable union of finite sets, hence countable. $\square$

Problem 2.3. Prove that there exist real numbers which are not algebraic.

Solution (sketch). This follows from Problem 2.2, since $\mathbb{R}$ is not countable. $\square$

Problem 2.4. Is $\mathbb{R} \setminus \mathbb{Q}$ countable?

Solution (sketch). No. The set $\mathbb{Q}$ is countable and $\mathbb{R}$ is not countable. $\square$

Problem 2.5. Construct a bounded subset of $\mathbb{R}$ with exactly 3 limit points.

Solution (sketch). For example, the limit points of the set

$$E = \big\{ \tfrac{1}{n} \colon n \in \mathbb{Z}_{>0} \big\} \cup \big\{ 1 + \tfrac{1}{n} \colon n \in \mathbb{Z}_{>0} \big\} \cup \big\{ 2 + \tfrac{1}{n} \colon n \in \mathbb{Z}_{>0} \big\}$$

are 0, 1, and 2. $\square$

A correct solution includes a proof that 0, 1, and 2 are limit points of E, and also a proof that E has no other limit points.

Problem 2.6. For a metric space X and a subset $E \subset X$, let E' denote the set of limit points of E.

(1) Prove that E' is closed.
(2) Prove that $\big(\overline{E}\big)' = E'$.
(3) Is $(E')'$ always equal to E'?

Solution to (1). We claim that $(E')' \subset E'$. To prove the claim, let $x \in (E')'$ and let $\varepsilon > 0$. By the definition of $(E')'$, there is $y \in E' \cap (N_{\varepsilon}(x) \setminus \{x\})$. Define

$$\delta = \min\big(d(x, y), \, \varepsilon - d(x, y)\big).$$

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Then $\delta > 0$. By the definition of E', there is $z \in E \cap (N_\delta(y) \setminus \{y\})$. Then

$$d(z, y) < \delta \le d(x, y),$$

so $z \neq x$. Also,

$$d(z, x) \le d(z, y) + d(y, x) < \delta + d(y, x) \le (\varepsilon - d(x, y)) + d(x, y) = \varepsilon,$$

so $z \in N_\varepsilon(x)$. This completes the proof of the claim.

The claim implies that all limit points of E' are in E', so E' is closed. □

Here is a different way to prove that $(E')' \subset E'$.

Alternate solution to (1) (sketch). Let $x \in (E')'$ and let $\varepsilon > 0$. By definition, there is a point $y \in E' \cap (N_{\varepsilon/2}(x) \setminus \{x\})$. By Theorem 2.20 of Rudin, there are infinitely many points in $E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$. In particular there is a point $z \in E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$ with $z \neq x$. Now $z \in E \cap (N_\varepsilon(x) \setminus \{x\})$.

Finish as in the first solution. □

Solution to (2). We first claim that if A and B are any subsets of X, then $(A \cup B)' \subset A' \cup B'$. The fastest way to prove the claim is to assume that $x \in (A \cup B)'$ but $x \notin A'$, and to show that $x \in B'$. Accordingly, let $x \in (A \cup B)' \setminus A'$. Since $x \notin A'$, there is $\varepsilon_0 > 0$ such that $N_{\varepsilon_0}(x) \cap A \subset \{x\}$.

Now let $\varepsilon > 0$; we show that $N_\varepsilon(x) \cap B$ contains at least one point different from x. To do so, set $r = \min(\varepsilon, \varepsilon_0) > 0$. Because $x \in (A \cup B)'$, there is $y \in N_r(x) \cap (A \cup B)$ with $y \neq x$. Then $y \notin A$ because $r \le \varepsilon_0$. So necessarily $y \in B$. We have shown that $N_\varepsilon(x) \cap B$ contains at least one point, namely y, which is different from x.

The previous paragraph shows that $x \in B'$, and completes the proof of the claim.

We also claim that if $A \subset B \subset X$ then $A' \subset B'$. The proof of this claim is obvious, since if $N_\varepsilon(x) \cap (A \setminus \{x\}) \neq \varnothing$, then certainly $N_\varepsilon(x) \cap (B \setminus \{x\}) \neq \varnothing$.

We now prove that $\left(\overline{E}\right)' \subset E'$. Using the first claim at the second step and $(E')' \subset E'$ (proved in the proof of Part (2)) at the third step, we have

$$\left(\overline{E}\right)' = (E \cup E')' \subset E' \cup (E')' \subset E' \cup E' = E',$$

as desired. The inclusion $E' \subset \left(\overline{E}\right)'$ follows from the second claim. □

Alternate solution to (2) (sketch). An alternate proof that $\left(\overline{E}\right)' \subset E'$ can be obtained by slightly modifying either of the proofs above that $(E')' \subset E'$. □

Solution to (3). The answer is no. Take

$$E = \{0\} \cup \left\{ \tfrac{1}{n} : n \in \mathbb{Z}_{>0} \right\}.$$

Then $E' = \{0\}$ and $(E')' = \varnothing$. (Of course, you must prove these facts.) □

Problem 2.8. If $E \subset \mathbb{R}^2$ is open, is every point of E a limit point of E? What if E is closed instead of open?

Solution (sketch). Every point of an open set $E \subset \mathbb{R}^2$ is a limit point of E. Indeed, if $x \in E$, then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$, and it is easy to show that x is a limit point of $N_\varepsilon(x)$.

Not every point of a closed set need be a limit point. Take $E = \{(0, 0)\}$, which has no limit points. □

Warning: in a general metric space, it need *not* be true that every point in an open set is a limit point of that set.

Problem 2.9. For a metric space X and a subset $E \subset X$, let E° denote the set of interior points of a set E, that is, the interior of E.

(1) Prove that E° is open.
(2) Prove that E is open if and only if $E^\circ = E$.
(3) If G is open and $G \subset E$, prove that $G \subset E^\circ$.
(4) Prove that $X \setminus E^\circ = \overline{X \setminus E}$.
(5) Prove or disprove: $\left(\overline{E}\right)^\circ = E^\circ$.
(6) Prove or disprove: $\overline{E^\circ} = \overline{E}$.

Solution to (1). Let $x \in E^\circ$. Then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$.

We claim that $N_\varepsilon(x) \subset E^\circ$. To prove the claim, let $y \in N_\varepsilon(x)$. Since $N_\varepsilon(x)$ is open, there is $\delta > 0$ such that $N_\delta(y) \subset N_\varepsilon(x)$. So $N_\delta(y) \subset E$. This shows that $y \in E^\circ$, proving the claim.

We have shown that for every $x \in E^\circ$ there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E^\circ$. That is, E° is open. □

Solution to (2). If E is open, then $E = E^\circ$ by the definition of E°. If $E = E^\circ$, then E is open by Part (a). □

Solution to (3) (sketch). We first claim that if $A \subset B \subset X$, then $A^\circ \subset B^\circ$. To prove the claim, let $x \in A^\circ$. By definition, there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset A$. Then also $N_\varepsilon(x) \subset B$, so $x \in B^\circ$. The claim is proved.

Now let $G \subset X$ be an open set such that $G \subset E$. Using (2) at the first step and the claim at the second step, we have

$$G = G^\circ \subset E^\circ,$$

as desired. □

Solution to (4) (sketch). First show that $X \setminus E^\circ \subset \overline{X \setminus E}$. If $x \notin E$, then clearly $x \in \overline{X \setminus E}$. Otherwise, consider $x \in E \setminus E^\circ$. Rearranging the statement that x fails to be an interior point of E, and noting that x itself is not in $X \setminus E$, one gets exactly the statement that x is a limit point of $X \setminus E$.

Now show that $\overline{X \setminus E} \subset X \setminus E^\circ$. If $x \in X \setminus E$, then clearly $x \notin E^\circ$. If $x \notin X \setminus E$ but x is a limit point of $X \setminus E$, then one simply rearranges the definition of a limit point to show that x is not an interior point of E. □

Solution to (5) (sketch). This is false. Example: take $X = \mathbb{R}$ and $E = (0,1) \cup (1,2)$. We have $E^\circ = E$, $\overline{E} = [0,2]$, and $\left(\overline{E}\right)^\circ = (0,2)$. □

Alternate solution to (5) (sketch). This is false. Example: take $X = \mathbb{R}$ and $E = \mathbb{Q}$. We have $E^\circ = \varnothing$, $\overline{E} = \mathbb{R}$, and $\left(\overline{E}\right)^\circ = \mathbb{R}$. □

Solution to (6) (sketch). This is false. Example: take $X = \mathbb{R}$ and $E = (0,1) \cup \{2\}$. Then $\overline{E} = [0,1] \cup \{2\}$, $E^\circ = (0,1)$, and $\overline{E^\circ} = [0,1]$. □

Alternate solution to (6) (sketch). This is false. Example: take $X = \mathbb{R}$ and $E = \mathbb{Q}$. Then $\overline{E} = \mathbb{R}$, $E^\circ = \varnothing$, and $\overline{E^\circ} = \varnothing$. □

Second alternate solution to (6) (sketch). This is false. Example: take $X = \mathbb{R}$ and $E = \{0\}$. Then $\overline{E} = \{0\}$, $E^\circ = \varnothing$, and $\overline{E^\circ} = \varnothing$. □

Problem 2.11. Which of the following are metrics on $\mathbb{R}$?

(1) $d_1(x, y) = (x - y)^2$ for $x, y \in \mathbb{R}$.
(2) $d_2(x, y) = \sqrt{|x - y|}$ for $x, y \in \mathbb{R}$.
(3) $d_3(x, y) = |x^2 - y^2|$ for $x, y \in \mathbb{R}$.
(4) $d_4(x, y) = |x - 2y|$ for $x, y \in \mathbb{R}$.
(5) $d_5(x, y) = \dfrac{|x - y|}{1 + |x - y|}$ for $x, y \in \mathbb{R}$.

Solution to (1) (sketch). No. The triangle inequality fails with $x = 0$, $y = 2$, and $z = 4$. □

Solution to (2) (sketch). Yes. Some work is needed to check the triangle inequality. □

Solution to (3) (sketch). No. We have $d_3(1, -1) = 0$. □

Solution to (4) (sketch). No. We have $d_4(1, 1) \neq 0$. □

Alternate solution to (4) (sketch). No. We have $d_4(1, 6) \neq d_4(6, 1)$. □

Solution to (5). Yes.

It is obvious that $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and that $d(x, y) = 0$ if and only if $x = y$. It is also obvious that $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}$.

It remains to prove the triangle inequality.

We first claim that the function $t \mapsto \frac{t}{1+t}$ is nondecreasing on $[0, \infty)$. To prove the claim, let $s, t \in \mathbb{R}$ satisfy $0 \leq s \leq t$. Then

$$\frac{t}{1+t} - \frac{s}{1+s} = \frac{t(1+s) - s(1+t)}{(1+t)(1+s)} = \frac{t-s}{(1+t)(1+s)} \geq 0.$$

This proves the claim.

We next claim that if $a, b \in [0, \infty)$ then

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

To prove the claim, we calculate:

$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

The claim is proved.

Now let $x, y, z \in \mathbb{R}$. Then, using $|x - z| \leq |x - y| + |y - z|$ and the first claim at the second step and using the second claim at the third step,

$$\begin{aligned} d_5(x, z) = \frac{|x - z|}{1 + |x - z|} &\leq \frac{|x - y| + |y - z|}{1 + |x - y| + |y - z|} \\ &\leq \frac{|x - y|}{1 + |x - y|} + \frac{|y - z|}{1 + |y - z|} = d_5(x, y) + d_5(y, z). \end{aligned}$$

This completes the solution. □

The proof given is easier than what most people did the last time I assigned this problem.

The first claim can also be proved using elementary calculus. This method isn't really legitimate because we haven't done any calculus yet.