MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 3

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.14: Give an example of an open cover of the interval $(0, 1) \subset \mathbb{R}$ which has no finite subcover.

Solution (sketch). Use $\{(1/n, 1) \colon n \in \mathbb{Z}_{>0}\}$. (You must show that this works.) $\square$

Problem 2.16: Regard $\mathbb{Q}$ as a metric space with the usual metric. Let $E = \{x \in \mathbb{Q} \colon 2 < x^2 < 3\}$. Prove that E is a closed and bounded subset of $\mathbb{Q}$ which is not compact. Is E an open subset of $\mathbb{Q}$?

Solution. Clearly E is bounded.

We claim that E is closed. We have

$$\mathbb{Q} \setminus E = \mathbb{Q} \cap \big[\big(-\infty, -\sqrt{3}\big) \cup \big(-\sqrt{2}, \sqrt{2}\big) \cup \big(\sqrt{3}, \infty\big) \big],$$

so $\mathbb{Q} \setminus E$ is open by Theorem 2.30. The claim follows.

Theorem 2.23 implies that E is not compact, because it is a subset of $\mathbb{R}$ which is not closed in $\mathbb{R}$.

To see that E is open in $\mathbb{Q}$, write

$$E = \mathbb{Q} \cap \big[\big(-\sqrt{3}, -\sqrt{2}\big) \cup \big(\sqrt{2}, \sqrt{3}\big) \big],$$

which is open by Theorem 2.30. $\square$

The following alternate solution uses different, direct, proofs of all parts except boundedness. It is also correct to give slick proofs as in the first solution for some parts and direct proofs as below for the others.

Alternate solution (sketch). Clearly E is bounded.

We claim that E is closed. Suppose $x \in \mathbb{Q}$ is a limit point of E which is not in E. Since we can't have $x^2 = 2$ or $x^2 = 3$, we must have $x^2 < 2$ or $x^2 > 3$. Assume $x^2 > 3$. (The other case is handled similarly.) Let $r = |x| - \sqrt{3} > 0$. Then every point $z \in N_r(x)$ satisfies

$$|z| \geq |x| - |x - z| > |x| - r > 0,$$

which implies that $z^2 > (|x| - r)^2 = 3$. This shows that $z \notin E$, which contradicts the assumption that x is a limit point of E. The claim is proved.

The set E is not compact because, for example, the sets

$$\big\{ y \in \mathbb{Q} \colon 2 + \tfrac{1}{n} < y^2 < 3 - \tfrac{1}{n} \big\}$$

form an open cover of E which has no finite subcover. (Fill in the details!)

A direct proof that E is open is fairly easy, but is omitted. $\square$

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Problem 2.19: Let X be a metric space, fixed throughout this problem.

(1) If A and B are disjoint closed subsets of X, prove that they are separated.
(2) If A and B are disjoint open subsets of X, prove that they are separated.
(3) Fix $x_0 \in X$ and $\delta > 0$. Set

$$A = \{x \in X : d(x, x_0) < \delta\} \qquad \text{and} \qquad B = \{x \in X : d(x, x_0) > \delta\}.$$

Prove that A and B are separated.
(4) Prove that if X is connected and contains at least two points, then X is uncountable.

Solution to (1). We have $A \cap \overline{B} = \overline{A} \cap B = A \cap B = \varnothing$ because A and B are closed. □

Solution to (2). The set $X \setminus A$ is a closed subset which contains B, and hence contains $\overline{B}$. Thus $A \cap \overline{B} = \varnothing$. Interchanging A and B, it follows that $\overline{A} \cap B = \varnothing$. □

Solution to (3) *(sketch).* Both A and B are open sets (proof!), and they are disjoint. So this follows from Part (2). □

Solution to (4). Let x and y be distinct points of X. Define $R = d(x, y)$. Then $R > 0$. For each $r \in (0, R)$, consider the sets

$$A_r = \{z \in X : d(z, x) < r\} \qquad \text{and} \qquad B_r = \{z \in X : d(z, x) > r\}.$$

They are separated by Part (3). They are not empty, since $x \in A_r$ and $y \in B_r$. Since X is connected, there must be a point $z_r \in X \setminus (A_r \cup B_r)$. Then $d(x, z_r) = r$.

If $r \neq s$, then $d(x, z_r) \neq d(x, z_s)$, so $z_r \neq z_s$. Thus $r \mapsto z_r$ defines an injective map from $(0, R)$ to X. Since $(0, R)$ is not countable, X can't be countable either. □

Problem 2.20: Let X be a metric space, and let $E \subset X$ be a connected subset. Is $\overline{E}$ necessarily connected? Is $\operatorname{int}(E)$ necessarily connected?

Solution to the first question. If E is connected, then $\overline{E}$ is necessarily connected. To prove this using Rudin's definition, assume $\overline{E} = A \cup B$ for separated sets A and B; we prove that one of A and B is empty. The sets $A_0 = A \cap E$ and $B_0 = B \cap E$ are separated sets such that $E = A_0 \cup B_0$. (They are separated because $\overline{A_0} \subset \overline{A}$ and $\overline{B_0} \subset \overline{B}$.) Because E is connected, one of A_0 and B_0 must be empty; without loss of generality, $A_0 = \varnothing$. Then $A \subset \overline{E} \setminus E$. Therefore $E \subset B$. But then $A \subset \overline{E} \subset \overline{B}$. Because A and B are separated, this can only happen if $A = \varnothing$. □

Alternate solution to the first question. If E is connected, we prove that $\overline{E}$ is necessarily connected, using the traditional definition. Thus, assume that $\overline{E} = A \cup B$ for disjoint relatively open sets A and B; we prove that one of A and B is empty. The sets $A_0 = A \cap E$ and $B_0 = B \cap E$ are disjoint relatively open sets in E such that $E = A_0 \cup B_0$. Because E is connected, one of A_0 and B_0 must be empty; without loss of generality, $A_0 = \varnothing$. Then $A \subset \overline{E} \setminus E$ and is relatively open in $\overline{E}$.

We claim that $A = \varnothing$. If not, let $x \in A$. Then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \cap \overline{E} \subset A$. So $N_\varepsilon(x) \cap \overline{E} \subset \overline{E} \setminus E$, which implies that $N_\varepsilon(x) \cap E = \varnothing$. This contradicts the fact that $x \in \overline{E}$, and proves the claim. □

Solution to the second question (sketch). The set $\operatorname{int}(E)$ need not be connected. The easiest example to write down is to take $X = \mathbb{R}^2$ and

$$E = \big\{ x \in \mathbb{R}^2 \colon \|x - (1,0)\| \leq 1 \big\} \cup \big\{ x \in \mathbb{R}^2 \colon \|x - (-1,0)\| \leq 1 \big\}.$$

Then

$$\operatorname{int}(E) = \big\{ x \in \mathbb{R}^2 \colon \|x - (1,0)\| < 1 \big\} \cup \big\{ x \in \mathbb{R}^2 \colon \|x - (-1,0)\| < 1 \big\}.$$

This set fails to be connected because the point $(0,0)$ is missing. A more dramatic example is two closed disks joined by a line, say

$$\begin{aligned} E = & \big\{ x \in \mathbb{R}^2 \colon \|x - (2,0)\| \leq 1 \big\} \cup \big\{ x \in \mathbb{R}^2 \colon \|x - (-2,0)\| \leq 1 \big\} \\ & \cup \big\{ (\alpha, 0) \in \mathbb{R}^2 \colon -3 \leq \alpha \leq 3 \big\}. \end{aligned}$$

Then

$$\operatorname{int}(E) = \big\{ x \in \mathbb{R}^2 \colon \|x - (2,0)\| < 1 \big\} \cup \big\{ x \in \mathbb{R}^2 \colon \|x - (-2,0)\| < 1 \big\},$$

which is not connected. □

Problem 2.22: Prove that $\mathbb{R}^n$ is separable.

Solution (sketch). The subset $\mathbb{Q}^n$ is countable by Theorem 2.13. To show that $\mathbb{Q}^n$ is dense, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $\varepsilon > 0$. Choose $y_1, \ldots, y_n \in \mathbb{Q}$ such that $|y_k - x_k| < \frac{\varepsilon}{n}$ for all k. (Why is this possible?) Then $y = (y_1, \ldots, y_n) \in \mathbb{Q}^n \cap N_\varepsilon(x)$. □

Problem 2.23: Prove that every separable metric space has a countable base.

Solution. Let X be a separable metric space. Let $S \subset X$ be a countable dense subset of X. Define

$$\mathcal{B} = \big\{ N_{1/n}(s) \colon s \in S, \, n \in \mathbb{Z}_{>0} \big\}.$$

Since $\mathbb{Z}_{>0}$ and S are countable, $\mathcal{B}$ is a countable collection of open subsets of X.

Now let $U \subset X$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since S is dense in X, the set $S \cap N_{1/n}(x)$ is nonempty. That is, there is $s \in S$ such that $d(s, x) < \frac{1}{n}$. Then $x \in N_{1/n}(s)$ and $N_{1/n}(s) \in \mathcal{B}$. It remains to show that $N_{1/n}(s) \subset U$. So let $y \in N_{1/n}(s)$. Then

$$d(x, y) \leq d(x, s) + d(s, y) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $y \in N_\varepsilon(x) \subset U$. □

Problem 2.25: Let K be a compact metric space. Prove that K has a countable base, and that K is separable.

The easiest way to do this is actually to prove first that K is separable, and then to use Problem 2.23. However, the direct proof that K has a countable base is not very different, so we give it here. We actually give two versions of the proof, which differ primarily in how the indexing is done. The first version is easier to write down correctly, but the second has the advantage of eliminating some of the subscripts, which can be important in more complicated situations. The second proof is shorter, even after the parenthetical remarks about indexing are deleted from the first proof. Afterwards, we give a direct proof that every metric space with a countable base is separable.

Solution. We prove that K has a countable base. For each $n \in \mathbb{Z}_{>0}$, the open sets $N_{1/n}(x)$, for $x \in K$, form an open cover of K. Since K is compact, this open cover has a finite subcover, say

$$\big\{N_{1/n}(x_{n,1}), N_{1/n}(x_{n,2}), \ldots, N_{1/n}(x_{n,k_n})\big\}$$

for suitable $x_{n,1}, x_{n,2}, \ldots, x_{n,k_n} \in K$. (Caution: for each n, the collection of elements used is different; therefore, they must be labelled independently by both n and a second parameter. The number of them also depends on n, so must be called k_n, $k(n)$, or something similar.)

Now let

$$\mathcal{B} = \big\{N_{1/n}(x_{n,j}) \colon n \in \mathbb{Z}_{>0},\ 1 \leq j \leq k_n\big\}.$$

(Both subscripts are used here.) Then $\mathcal{B}$ is a countable union of finite sets, hence countable. We show that $\mathcal{B}$ is a base for K.

Let $U \subset K$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since the sets

$$N_{1/n}(x_{n,1}),\ N_{1/n}(x_{n,2}),\ \ldots,\ N_{1/n}(x_{n,k_n})$$

cover K, there is $j \in \{1, 2, \ldots, k_n\}$ such that $x \in N_{1/n}(x_{n,j})$. (Here we see why the double indexing is necessary: the list of centers to choose from *depends on* n, and therefore their names must also depend on n.) By definition, $N_{1/n}(x_{n,j}) \in \mathcal{B}$. It remains to show that $N_{1/n}(x_{n,j}) \subset U$. So let $y \in N_{1/n}(x_{n,j})$. Since x and y are both in $N_{1/n}(x_{n,j})$, we have

$$d(x, y) \leq d(x, x_{n,j}) + d(x_{n,j}, y) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $y \in N_\varepsilon(x) \subset U$. We have proved that K has a countable base. $\square$

Alternate solution. We again prove that K has a countable base. For each $n \in \mathbb{Z}_{>0}$, the open sets $N_{1/n}(x)$, for $x \in K$, form an open cover of K. Since K is compact, this open cover has a finite subcover. That is, there is a finite set $F_n \subset K$ such that the sets $N_{1/n}(x)$, for $x \in F_n$, still cover K. Now let

$$\mathcal{B} = \big\{N_{1/n}(x) \colon n \in \mathbb{Z}_{>0},\ x \in F_n\big\}.$$

Then $\mathcal{B}$ is a countable union of finite sets, hence countable. We show that $\mathcal{B}$ is a base for K.

Let $U \subset K$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since the sets $N_{1/n}(y)$, for $y \in F_n$, cover K, there is $y \in F_n$ such that $x \in N_{1/n}(y)$. By definition, $N_{1/n}(y) \in \mathcal{B}$. It remains to show that $N_{1/n}(y) \subset U$. So let $z \in N_{1/n}(y)$. Since x and z are both in $N_{1/n}(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $z \in N_\varepsilon(x) \subset U$. We have proved that K has a countable base. $\square$

It remains to prove the following lemma.

Lemma 1. Let X be a metric space with a countable base. Then X is separable.

Proof. Let $\mathcal{B}$ be a countable base for X. Without loss of generality, we may assume $\varnothing \notin \mathcal{B}$. For each $U \in \mathcal{B}$, choose an element $x_U \in U$. Let $S = \{x_U \colon U \in \mathcal{B}\}$. Clearly S is (at most) countable. We show it is dense. So let $x \in X$. If $x \in S$, there is nothing to prove. Otherwise, we claim that x is a limit point of S. Let $\varepsilon > 0$ be arbitrary. Then $N_\varepsilon(x)$ is an open set in X, so there exists $U \in \mathcal{B}$ such that $x \in U \subset N_{\varepsilon}(x)$. In particular, $x_U \in N_{\varepsilon}(x)$. Since $x_U \neq x$ and since $\varepsilon > 0$ is arbitrary, this shows that x is a limit point of S. The claim is proved. $\square$

It is not in general true that every point of X is a limit point of S. If X is finite, or is a countable set with the discrete metric, then S will be equal to X, but X has no limit points.