MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 5

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.7: Let $a_n \geq 0$ for $n \in \mathbb{Z}_{>0}$. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{a_n}$ converges.

Solution. For all $x, y \in \mathbb{R}$, we have $(x-y)^2 \geq 0$. Multiplying out and rearranging gives $xy \leq \frac{1}{2}(x^2+y^2)$. For $n \in \mathbb{Z}_{>0}$, put $x = \sqrt{a_n}$ and $y = \frac{1}{n}$ in this inequality, getting the second part of the inequality

$$0 \leq \frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right).$$

Since both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge, the series

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$$

converges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{a_n}$ converges by the comparison test. □

Problem 3.8: Let $(b_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a bounded monotone sequence in $\mathbb{R}$, and let $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution (sketch). We first reduce to the case $\lim_{n\to\infty} b_n = 0$. Since $(b_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a bounded monotone sequence, it follows that $b = \lim_{n\to\infty} b_n$ exists. Set $c_n = b_n - b$. Then $(c_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a bounded monotone sequence with $\lim_{n\to\infty} c_n = 0$. Since $a_n b_n = a_n c_n + a_n b$ and $\sum_{n=1}^{\infty} a_n b$ converges, it suffices to prove that $\sum_{n=1}^{\infty} a_n c_n$ converges. That is, we may assume that $\lim_{n\to\infty} b_n = 0$.

With this assumption, if $b_1 \geq 0$, then $b_1 \geq b_2 \geq \cdots \geq 0$, so $\sum_{n=1}^{\infty} a_n b_n$ converges by Theorem 3.42 in the book. Otherwise, replace b_n by $-b_n$. □

Comment: Suppose $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ and $(b_n)_{n\in\mathbb{Z}_{\geq 0}}$ are sequences in $\mathbb{C}$ (or $\mathbb{R}$). It is certainly true that, for $m, n \in \mathbb{Z}_{>0}$ with $n > m$, we have

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \sum_{k=m}^{n} |a_k||b_k|.$$

However, the inequality

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \left(\left| \sum_{k=m}^{n} a_k \right| \right) \left(\max_{m \leq k \leq n} |b_k| \right)$$

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is not correct, not even if $b_k > 0$ for all $k \in \mathbb{Z}_{>0}$. Example: for $k \in \mathbb{Z}_{>0}$ set

$$a_k = (-1)^k \qquad \text{and} \qquad b_k = \frac{1}{k}.$$

Take $m = 1$, and let $n \in \mathbb{Z}_{>0}$ be any positive even integer. Then

$$\left| \sum_{k=m}^{n} a_k b_k \right| = \left| \sum_{k=1}^{n} \frac{(-1)^k}{k} \right| > 0 \qquad \text{and} \qquad \left(\left| \sum_{k=m}^{n} a_k \right| \right) \left(\max_{m \leq k \leq n} |b_k| \right) = 0 \cdot 1 = 0.$$

The inequality

$$\cancel{\left| \sum_{k=m}^{n} a_k b_k \right| \leq \left(\left| \sum_{k=m}^{n} a_k \right| \right) |b_k|}$$

doesn't make sense without specifying a value of k. Even then, it should be less confusingly written: for some specified r,

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \left(\left| \sum_{k=m}^{n} a_k \right| \right) |b_r|,$$

and whether it is true or not depends on the sequences $(a_n)_{n \in \mathbb{Z}_{\geq 0}}$ and $(b_n)_{n \in \mathbb{Z}_{\geq 0}}$ and the choice of r.