MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 6

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.9: Find the radius of convergence of each of the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} n^3 z^n$.

Solution 1. Use Theorem 3.20(c) of Rudin in the second step to get

$$\limsup_{n\to\infty} \sqrt[n]{n^3} = \left(\lim_{n\to\infty} \sqrt[n]{n}\right)^3 = 1.$$

It now follows from Theorem 3.39 of Rudin that the radius of convergence is 1. □

Solution 2. We show that the series converges for $|z| < 1$ and diverges for $|z| > 1$.

For $|z| = 0$, convergence is trivial.

If $0 < |z| < 1$, we use the Ratio Test (Theorem 3.34 of Rudin). We have

$$\lim_{n\to\infty} \frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} = \lim_{n\to\infty} |z| \left(\frac{n+1}{n}\right)^3 = |z| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3$$
$$= |z| \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^3 = |z|.$$

For $|z| < 1$ the hypotheses of Theorem 3.34(a) of Rudin are therefore satisfied, so that the series converges.

For $|z| > 1$, we use the Ratio Test. The same calculation as in the case $0 < |z| < 1$ gives

$$\lim_{n\to\infty} \frac{|(n+1)^3 z^{n+1}|}{n^3 z^n|} = |z|.$$

Since $|z| > 1$, it follows that there is N such that for all $n \geq N$ we have

$$\left| \frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} - |z| \right| < \tfrac{1}{2}(|z| - 1).$$

In particular,

$$\frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} > 1$$

for $n \geq N$. The hypotheses of Theorem 3.34(b) of Rudin are therefore satisfied, so that the series diverges.

Theorem 3.39 of Rudin implies that there is a unique $R \in [0, \infty]$ such that the series converges for $|z| < R$ and diverges for $|z| > R$. The results above imply that $R = 1$. □

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Solution 3. We calculate

$$\lim_{n\to\infty} \frac{|(n+1)^3|}{|n^3|} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^3 = 1.$$

According to Theorem 3.37 of Rudin, we have

$$\liminf_{n\to\infty} \frac{|(n+1)^3|}{|n^3|} \leq \liminf_{n\to\infty} \sqrt[n]{|n^3|} \leq \limsup_{n\to\infty} \sqrt[n]{|n^3|} \leq \limsup_{n\to\infty} \frac{|(n+1)^3|}{|n^3|}.$$

Therefore $\lim_{n\to\infty} \sqrt[n]{|n^3|}$ exists and is equal to 1. It now follows from Theorem 3.39 of Rudin that the radius of convergence is 1. □

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot z^n.$

Solution (sketch). Use the Ratio Test to show that the series converges for all z. (See Solution 2 to Part (a).) So the radius of convergence is ∞. □

Remark: Note that $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot z^n = e^{2z}.$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^2} \cdot z^n.$

Solution (sketch). Either the root or Ratio Test gives radius of convergence equal to $\frac{1}{2}$. (Use the methods of any of the three solutions to Part (a).) □

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n} \cdot z^n.$

Solution (sketch). Either the root or Ratio Test gives radius of convergence equal to 3. (Use the methods of any of the three solutions to Part (a).) □

Problem 3.16: Fix $\alpha > 0$. Choose $x_1 > \sqrt{\alpha}$, and recursively define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

for $n \in \mathbb{Z}_{>0}$.

(a) Prove that $(x_n)_{n\in\mathbb{Z}_{>0}}$ is nonincreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

Solution (sketch). Using the inequality $a^2 + b^2 \geq 2ab$, and assuming $x_n > 0$, we get

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \sqrt{x_n} \cdot \sqrt{\frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

It follows by induction that $x_n > \sqrt{\alpha}$ for all $n \in \mathbb{Z}_{>0}$. Next, for $n \in \mathbb{Z}_{>0}$ we have

$$x_n - x_{n+1} = \frac{x_n^2 - \alpha}{2x_n} > 0.$$

Thus $(x_n)_{n\in\mathbb{Z}_{>0}}$ is nonincreasing. We already know that this sequence is bounded below (by $\sqrt{\alpha}$), so $x = \lim_{n\to\infty} x_n$ exists. Letting $n \to \infty$ in the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

gives

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right).$$

This equation implies $x = \pm\sqrt{\alpha}$, and we must have $x = \sqrt{\alpha}$ because $(x_n)_{n\in\mathbb{Z}_{>0}}$ is bounded below by $\sqrt{\alpha} > 0$. □

(b) Set $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}.$$

Further show that, with $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

Solution (sketch). Prove all the relations at once by induction on n, together with the statement $\varepsilon_{n+1} > 0$. For $n = 1$, the relation

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$$

is just algebra, the inequality

$$\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

follows from $x_1 > \sqrt{\alpha}$, and the inequality

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

is just a rewritten form of

$$\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

The statement $\varepsilon_{n+1} > 0$ is clear from

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$$

and $x_1 > 0$.

Now assume all this is known for some value of n. As before, the relation

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$$

is just algebra, and implies that $\varepsilon_{n+1} > 0$. (We know that $x_n = \sqrt{\alpha} + \varepsilon_n > \sqrt{\alpha} > 0$.) Since $\varepsilon_n > 0$, we have $x_n > \sqrt{\alpha}$, so the inequality

$$\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

follows. To get the other inequality, write

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} = \frac{\varepsilon_n^2}{\beta} < \left(\frac{1}{\beta}\right)\cdot\left[\beta\cdot\left(\frac{\varepsilon_1}{\beta}\right)^{2^{n-1}}\right]^2 = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

□

(c) Specifically take $\alpha = 3$ and $x_1 = 2$. show that

$$\frac{\varepsilon_1}{\beta} < \frac{1}{10}, \qquad \varepsilon_5 < 4 \cdot 10^{-16}, \qquad \text{and} \qquad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution (sketch). This is just calculation. □

Problem 3.23: Let X be a metric space, and let $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$ be Cauchy sequences in X. Prove that $\lim_{n \to \infty} d(x_n, y_n)$ exists.

Solution. Since $\mathbb{R}$ is complete, it suffices to show that $(d(x_n, y_n))_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence. Let $\varepsilon > 0$. Choose N so large that if $m, n \geq N$, then both $d(x_m, x_n) < \frac{\varepsilon}{2}$ and $d(y_m, y_n) < \frac{\varepsilon}{2}$. For such m and n, we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ &< \frac{\varepsilon}{2} + d(x_m, y_m) + \frac{\varepsilon}{2} = d(x_m, y_m) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} d(x_m, y_m) &\leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \\ &< \frac{\varepsilon}{2} + d(x_n, y_n) + \frac{\varepsilon}{2} = d(x_n, y_n) + \varepsilon, \end{aligned}$$

so

$$|d(x_m, y_m) - d(x_n, y_n)| < \varepsilon.$$

This completes the solution. □

Problem 3.24: Let X be a metric space.

(a) Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$ be Cauchy sequences in X. We say they are equivalent, and write $(x_n)_{n \in \mathbb{Z}_{>0}} \sim (y_n)_{n \in \mathbb{Z}_{>0}}$, if $\lim_{n \to \infty} d(x_n, y_n) = 0$. Prove that this is an equivalence relation.

Solution. That $(x_n)_{n \in \mathbb{Z}_{>0}} \sim (x_n)_{n \in \mathbb{Z}_{>0}}$, and that $(x_n)_{n \in \mathbb{Z}_{>0}} \sim (y_n)_{n \in \mathbb{Z}_{>0}}$ implies $(y_n)_{n \in \mathbb{Z}_{>0}} \sim (x_n)_{n \in \mathbb{Z}_{>0}}$, are obvious. For transitivity, assume $(x_n)_{n \in \mathbb{Z}_{>0}} \sim (y_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}} \sim (z_n)_{n \in \mathbb{Z}_{>0}}$. For all $n \in \mathbb{Z}_{>0}$, we have

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n),$$

so

$$0 \leq \liminf_{n \to \infty} d(x_n, z_n) \leq \limsup_{n \to \infty} d(x_n, z_n) \leq \lim_{n \to \infty} d(x_n, y_n) + \lim_{n \to \infty} d(y_n, z_n) = 0.$$

Therefore $\lim_{n \to \infty} d(x_n, z_n) = 0$, as desired. □

(b) Let X^* be the set of equivalence classes from Part (a). Denote by $[(x_n)_{n \in \mathbb{Z}_{>0}}]$ the equivalence class in X^* of the Cauchy sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$. If $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$ are Cauchy sequences in X, set

$$\Delta_0\big((x_n)_{n \in \mathbb{Z}_{>0}}, (y_n)_{n \in \mathbb{Z}_{>0}}\big) = \lim_{n \to \infty} d(x_n, y_n).$$

Prove that $\Delta_0\big((x_n)_{n \in \mathbb{Z}_{>0}}, (y_n)_{n \in \mathbb{Z}_{>0}}\big)$ only depends on $[(x_n)_{n \in \mathbb{Z}_{>0}}]$ and $[(y_n)_{n \in \mathbb{Z}_{>0}}]$. Moreover, show that the formula

$$\Delta\big([(x_n)_{n \in \mathbb{Z}_{>0}}], [(y_n)_{n \in \mathbb{Z}_{>0}}]\big) = \Delta_0\big((x_n)_{n \in \mathbb{Z}_{>0}}, (y_n)_{n \in \mathbb{Z}_{>0}}\big)$$

defines a metric on X^*.

Solution (sketch). It is easy to check that Δ_0 is a semimetric, that is, it satisfies all the conditions for a metric except that possibly

$$\Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big) = 0$$

without having $(x_n)_{n\in\mathbb{Z}_{>0}} = (y_n)_{n\in\mathbb{Z}_{>0}}$. (For example,

$$\begin{aligned}
&\Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (z_n)_{n\in\mathbb{Z}_{>0}}\big)\\
&\qquad = \lim_{n\to\infty} d(x_n, z_n) \le \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n)\\
&\qquad = \Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big) + \Delta_0\big((y_n)_{n\in\mathbb{Z}_{>0}},\, (z_n)_{n\in\mathbb{Z}_{>0}}\big)
\end{aligned}$$

because for all $n \in \mathbb{Z}_{>0}$ we have $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$; the other properties are proved similarly.) Further, by definition, $(x_n)_{n\in\mathbb{Z}_{>0}} \sim (y_n)_{n\in\mathbb{Z}_{>0}}$ if and only if $\Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big) = 0$.

Now we prove that $\Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big)$ only depends on

$$[(x_n)_{n\in\mathbb{Z}_{>0}}] \qquad \text{and} \qquad [(y_n)_{n\in\mathbb{Z}_{>0}}].$$

Suppose that $(x_n)_{n\in\mathbb{Z}_{>0}} \sim (r_n)_{n\in\mathbb{Z}_{>0}}$ and $(y_n)_{n\in\mathbb{Z}_{>0}} \sim (s_n)_{n\in\mathbb{Z}_{>0}}$. Then, by the previous paragraph,

$$\begin{aligned}
&\Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big)\\
&\le \Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (r_n)_{n\in\mathbb{Z}_{>0}}\big)\\
&\qquad + \Delta_0\big((r_n)_{n\in\mathbb{Z}_{>0}},\, (s_n)_{n\in\mathbb{Z}_{>0}}\big) + \Delta_0\big((s_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big)\\
&= 0 + \Delta_0\big((r_n)_{n\in\mathbb{Z}_{>0}},\, (s_n)_{n\in\mathbb{Z}_{>0}}\big) + 0\\
&= \Delta_0\big((r_n)_{n\in\mathbb{Z}_{>0}},\, (s_n)_{n\in\mathbb{Z}_{>0}}\big);
\end{aligned}$$

similarly

$$\Delta_0\big((r_n)_{n\in\mathbb{Z}_{>0}},\, (s_n)_{n\in\mathbb{Z}_{>0}}\big) \le \Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big).$$

Thus

$$\Delta_0\big((r_n)_{n\in\mathbb{Z}_{>0}},\, (s_n)_{n\in\mathbb{Z}_{>0}}\big) = \Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big).$$

The previous paragraph implies that Δ is well defined. It is now easy to check that Δ satisfies all the conditions for a metric except that possibly

$$\Delta\big([(x_n)_{n\in\mathbb{Z}_{>0}}], [(y_n)_{n\in\mathbb{Z}_{>0}}]\big) = 0$$

without having $[(x_n)_{n\in\mathbb{Z}_{>0}}] = [(y_n)_{n\in\mathbb{Z}_{>0}}]$. (For example,

$$\begin{aligned}
&\Delta\big([(x_n)_{n\in\mathbb{Z}_{>0}}],\, [(z_n)_{n\in\mathbb{Z}_{>0}}]\big)\\
&\qquad = \Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (z_n)_{n\in\mathbb{Z}_{>0}}\big)\\
&\qquad \le \Delta_0\big((x_n)_{n\in\mathbb{Z}_{>0}},\, (y_n)_{n\in\mathbb{Z}_{>0}}\big) + \Delta_0\big((y_n)_{n\in\mathbb{Z}_{>0}},\, (z_n)_{n\in\mathbb{Z}_{>0}}\big)\\
&\qquad = \Delta\big([(x_n)_{n\in\mathbb{Z}_{>0}}],\, [(y_n)_{n\in\mathbb{Z}_{>0}}]\big) + \Delta_0\big([(y_n)_{n\in\mathbb{Z}_{>0}}],\, [(z_n)_{n\in\mathbb{Z}_{>0}}]\big);
\end{aligned}$$

the other properties are proved similarly.)

Finally, if $\Delta\big([(x_n)_{n\in\mathbb{Z}_{>0}}],\, [(y_n)_{n\in\mathbb{Z}_{>0}}]\big) = 0$ then it follows from the definition of $(x_n)_{n\in\mathbb{Z}_{>0}} \sim (y_n)_{n\in\mathbb{Z}_{>0}}$ that we actually do have $[(x_n)_{n\in\mathbb{Z}_{>0}}] = [(y_n)_{n\in\mathbb{Z}_{>0}}]$. So Δ is a metric. □

(c) Prove that X^* is complete in the metric Δ.

The basic idea is as follows. We start with a Cauchy sequence in X^*, which is a sequence of (equivalence classes of) Cauchy sequences in X. The limit is supposed to be (the equivalence class of) another Cauchy sequence in X. This sequence is constructed by taking suitable terms from the given sequences. The choices get a little messy. Afterwards, we will give a different proof, similar in length but with simpler notation.

Solution (sketch). Let $(a_k)_{k\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in X^*; we show that it converges. For each $k \in \mathbb{Z}_{>0}$, a_k is an equivalence class of Cauchy sequences in X. Therefore there is a Cauchy sequence $\big(x_n^{(k)}\big)_{n\in\mathbb{Z}_{>0}}$ in X such that $a_k = \big[\big(x_n^{(k)}\big)_{n\in\mathbb{Z}_{>0}}\big]$. The limit we construct in X^* will have the form $a = \big[\big(x_n^{(f(n))}\big)\big]$ for a suitable function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

We recursively construct strictly positive integers

$$M(1) < M(2) < M(3) < \cdots \qquad \text{and} \qquad N(1) < N(2) < N(3) < \cdots$$

such that:

(1) $\Delta(a_k, a_l) < 2^{-r-2}$ for $k,\, l \geq M(r)$.
(2) For all $k,\, l \leq M(r)$ and $m \geq N(r)$, we have $d\big(x_m^{(k)},\, x_m^{(l)}\big) < \Delta(a_k, a_l) + 2^{-r-2}$.
(3) For all $k \leq M(r)$ and $m,\, n \geq N(r)$, we have $d\big(x_m^{(k)},\, x_n^{(k)}\big) < 2^{-r-2}$.

To do this, first use the fact that $(a_k)_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence to find $M(1)$. Then choose $N(1)$ large enough to satisfy (2) and (3) for $r = 1$; this can be done because $\lim_{m\to\infty} d\big(x_m^{(k)},\, x_m^{(l)}\big) = \Delta(a_k, a_l)$ (for (2)) and because $\big(x_n^{(k)}\big)_{n\in\mathbb{Z}_{>0}}$ is Cauchy (for (3)), and using the fact that there are only finitely many pairs (k, l) to consider in (2) and only finitely many k to consider in (3). Next, use the fact that $(a_k)_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence to find $M(2)$, and also require $M(2) > M(1)$. Choose $N(2) > N(1)$ by the same reasoning as used to get $N(1)$. Proceed recursively.

We now take a to be the equivalence class of the sequence

$$x_1^{(1)},\, \ldots,\, x_{N(1)-1}^{(1)},\, x_{N(1)}^{(M(1))},\, \ldots,\, x_{N(2)-1}^{(M(1))},\, x_{N(2)}^{(M(2))},\, \ldots,\, x_{N(3)-1}^{(M(2))},\, x_{N(3)}^{(M(3))},\, \ldots.$$

That is, the function f above is given by $f(n) = M(r)$ for $N(r) \leq n \leq N(r+1) - 1$.

We show that $\big(x_n^{(f(n))}\big)_{n\in\mathbb{Z}_{>0}}$ is Cauchy. First, estimate (justifications afterwards):

$$\begin{aligned} d\left(x_{N(r)}^{(M(r))},\, x_{N(r+1)}^{(M(r+1))}\right) &\leq d\left(x_{N(r)}^{(M(r))},\, x_{N(r+1)}^{(M(r))}\right) + d\left(x_{N(r+1)}^{(M(r))},\, x_{N(r+1)}^{(M(r+1))}\right) \\ &< 2^{-r-2} + \Delta\big(a_{M(r)},\, a_{M(r+1)}\big) + 2^{-r-3} \\ &< 2^{-r-2} + 2^{-r-2} + 2^{-r-3} < 3 \cdot 2^{-r-2}. \end{aligned}$$

The first term on the second line is gotten from (2) above, because $M(r) \leq M(r)$ and $N(r),\, N(r+1) \geq N(r)$. The other two terms on the second line are gotten from (3) above (for $r+1$), because $M(r),\, M(r+1) \leq M(r+1)$ and $N(r+1) \geq N(r+1)$. The estimate used to get the third line comes from (1) above. Then use induction to show that $s \geq r$ implies

$$d\left(x_{N(r)}^{(M(r))},\, x_{N(s)}^{(M(s))}\right) \leq 3\big[2^{-r-2} + 2^{-r-3} + \cdots + 2^{-s-1}\big].$$

Now let $n \geq N(r)$ be arbitrary. Choose $s \geq r$ such that $N(s) \leq n \leq N(s+1) - 1$. Then $f(n) = M(s)$, so

$$d\left(x_n^{(f(n))},\, x_{N(s)}^{(M(s))}\right) = d\left(x_n^{(M(s))},\, x_{N(s)}^{(M(s))}\right) < 3 \cdot 2^{-s-2},$$

using (3) above with $r = s$ and $k = M(s)$. Therefore

$$d\left(x_{N(r)}^{(M(r))},\, x_n^{(f(n))}\right) < 3\big[2^{-r-2} + 2^{-r-3} + \cdots + 2^{-s-1} + 2^{-s-2}\big] < 3 \cdot 2^{-r-1}.$$

Finally, if $m,\, n \geq N(r)$ are arbitrary, then

$$d\left(x_m^{(f(m))},\, x_n^{(f(n))}\right) \leq d\left(x_m^{(f(m))},\, x_{N(r)}^{(M(r))}\right) + d\left(x_{N(r)}^{(M(r))},\, x_n^{(f(n))}\right) < 3 \cdot 2^{-r}.$$

This is enough to prove that $\big(x_n^{(f(n))}\big)_{n \in \mathbb{Z}_{>0}}$ is Cauchy.

It remains to show that $\Delta(a_k, a) \to 0$. Fix k, choose r with $M(r-1) < k \leq M(r)$, and let $n \geq N(r)$. From the previous paragraph we have

$$d\left(x_{N(r)}^{(M(r))},\, x_n^{(f(n))}\right) < 3 \cdot 2^{-r-1}.$$

Since $n,\, N(r) \geq N(r)$ and $k \leq M(r)$, condition (3) above gives

$$d\left(x_n^{(k)},\, x_{N(r)}^{(k)}\right) < 2^{-r-2}.$$

Furthermore,

$$d\left(x_{N(r)}^{(k)},\, x_{N(r)}^{(M(r))}\right) < \Delta(a_k,\, a_{M(r)}) + 2^{-r-2} < 2^{-r-1} + 2^{-r-2},$$

where the first step uses (2) above and the inequalities $k,\, M(r) \leq M(r)$ and $N(r) \geq N(r)$, while the second step uses (1) above and the inequality $r \geq M(r-1)$. Combining these estimates using the triangle inequality, we get

$$d\big(x_n^{(k)},\, x_n^{(f(n))}\big) < 2^{-r-2} + [2^{-r-1} + 2^{-r-2}] + 3 \cdot 2^{-r-1} < 2^{-r+2}.$$

Therefore

$$\Delta(a_k, a) = \lim_{n \to \infty} d\big(x_n^{(k)},\, x_n^{(f(n))}\big) \leq 2^{-r+2}$$

whenever $M(r-1) < k \leq M(r)$. Since $M(r) \to \infty$ as $r \to \infty$, this implies that $\Delta(a_k, a) \to 0$. □

Here is a perhaps slicker way to do the same thing, although it isn't any shorter. Essentially, by passing to suitable subsequences, we can take the representative of the limit to be the diagonal sequence, that is, $f(n) = n$ in the proof above. The construction requires the following lemmas.

Lemma 1. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in a metric space Y. Then there is a subsequence $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ of $(x_n)_{n \in \mathbb{Z}_{>0}}$ such that $d(x_{k(n+1)},\, x_{k(n)}) < 2^{-n}$ for all $n \in \mathbb{Z}_{>0}$.

Proof. Choose a sequence $(k(n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ recursively to satisfy $k(n+1) > k(n)$ and $d(x_l, x_m) < 2^{-n}$ for all $l,\, m \geq k(n)$. □

Lemma 2. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in a metric space Y. For $m, n \in \mathbb{Z}_{>0}$ with $n > m$, we have

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k,\, x_{k+1}).$$

Moreover, if $\sum_{k=1}^{\infty} d(x_k,\, x_{k+1})$ converges, then $(x_n)_{n \in \mathbb{Z}_{>0}}$ is Cauchy.

Proof. The first part is just the triangle inequality.

The Cauchy criterion for convergence of a series implies that for all $\varepsilon > 0$, there is N such that if $n > m \geq N$, then $\sum_{k=m}^{n-1} d(x_k, x_{k+1}) < \varepsilon$. But the first part gives $d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1})$. Thus if $n > m \geq N$ then $d(x_m, x_n) < \varepsilon$. The case $m > n \geq N$ is handled by symmetry, and the case $n = m \geq N$ is trivial. □

Lemma 3. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in a metric space Y, and let $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ be a subsequence. Then $\lim_{n \to \infty} d(x_n, x_{k(n)}) = 0$.

Proof. Let $\varepsilon > 0$. Choose N such that if $m, n \geq N$ then $d(x_m, x_n) < \varepsilon$. If $n \geq N$, then $k(n) \geq n \geq N$, so $d(x_n, x_{k(n)}) < \varepsilon$. □

Lemma 4. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in a metric space Y. If $(x_n)_{n \in \mathbb{Z}_{>0}}$ has a convergent subsequence, then $(x_n)_{n \in \mathbb{Z}_{>0}}$ converges.

Proof. Let $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ be a subsequence with limit x. Then, using Lemma 3,

$$d(x_n, x) \leq d(x_n, x_{k(n)}) + d(x_{k(n)}, x) \to 0.$$

This completes the proof. □

Proof of the result. Let (a_k) be a Cauchy sequence in X^*; we show that it converges. Use Lemma 1 to choose a subsequence $(a_{r(k)})_{k \in \mathbb{Z}_{>0}}$ such that

$$\Delta(a_{r(k)}, a_{r(k+1)}) < 2^{-k}$$

for all k. By Lemma 4, it suffices to show that $(a_{r(k)})_{k \in \mathbb{Z}_{>0}}$ converges. Without loss of generality, therefore, we may assume the original sequence $(a_k)_{k \in \mathbb{Z}_{>0}}$ satisfies $\Delta(a_k, a_{k+1}) < 2^{-k}$ for all k.

For each $k \in \mathbb{Z}_{>0}$, a_k is an equivalence class of Cauchy sequences in X. By Lemma 1 and Lemma 3, there is a sequence $\left(x_n^{(k)}\right)_{n \in \mathbb{Z}_{>0}}$ in X such that $a_k = \left[\left(x_n^{(k)}\right)_{n \in \mathbb{Z}_{>0}}\right]$ and $d\left(x_n^{(k)}, x_{n+1}^{(k)}\right) < 2^{-n}$ for all $n \in \mathbb{Z}_{>0}$.

We now estimate $d\left(x_n^{(k)}, x_n^{(k+1)}\right)$. For $\varepsilon > 0$, we can find $m > n$ such that

$$\begin{aligned} d\left(x_m^{(k)}, x_m^{(k+1)}\right) &< \Delta\left(\left[(x_n^{(k)})_{n \in \mathbb{Z}_{>0}}\right], \left[(x_n^{(k+1)})_{n \in \mathbb{Z}_{>0}}\right]\right) + \varepsilon \\ &= \Delta(a_k, a_{k+1}) + \varepsilon < 2^{-k} + \varepsilon. \end{aligned}$$

Now, using Lemma 2,

$$d(x_n^{(k)}, x_m^{(k)}) < 2^{-n} + 2^{-n-1} + \cdots + 2^{-m+1} < 2^{-n+1}.$$

The same estimate holds for $d\left(x_n^{(k+1)}, x_m^{(k+1)}\right)$. Therefore

$$\begin{aligned} d\left(x_n^{(k)}, x_n^{(k+1)}\right) &\leq d\left(x_n^{(k)}, x_m^{(k)}\right) + d\left(x_m^{(k)}, x_m^{(k+1)}\right) + d\left(x_m^{(k+1)}, x_n^{(k+1)}\right) \\ &< 2^{-n+1} + \varepsilon + 2^{-n+1}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives

$$d\left(x_n^{(k)}, x_n^{(k+1)}\right) \leq 2^{-n+2}$$

for all $n, k \in \mathbb{Z}_{>0}$.

Now define $y_n = x_n^{(n)}$ for $n \in \mathbb{Z}_{>0}$. First, observe that

$$\begin{aligned} d(y_n, y_{n+1}) &\leq d\left(x_n^{(n)}, x_{n+1}^{(n)}\right) + d\left(x_{n+1}^{(n)}, x_{n+1}^{(n+1)}\right) \\ &\leq 2^{-n} + 2^{-n+1} < 2^{-n+2}. \end{aligned}$$

Therefore $(y_n)_{n \in \mathbb{Z}_{>0}}$ is Cauchy, by Lemma 2. So $a = [(y_n)_{n \in \mathbb{Z}_{>0}}] \in X^*$.

It remains to show that $\Delta(a_n, a) \to 0$. If $m > n$, then we use the estimates $d\big(x_n^{(n)}, x_m^{(n)}\big) < 2^{-n+1}$ (as above) and $d(y_n, y_m) < 2^{-n+3}$ (obtained similarly, using Lemma 2 again) to get

$$d\big(x_m^{(n)}, y_m\big) \leq d\big(x_m^{(n)}, x_n^{(n)}\big) + d(y_n, y_m) < 2^{-n+4}.$$

In particular,

$$\Delta(a_n, a) = \lim_{m \to \infty} d\big(x_m^{(n)}, y_m\big) \leq 2^{-n+4}.$$

Thus $\Delta(a_n, a) \to 0$, as desired. □

(d) Define $f \colon X \to X^*$ by $f(x) = [(x, x, x, \dots)]$. Prove that f is isometric, that is, that $\Delta(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Solution (sketch). This is immediate. □

(e) Prove that $f(X)$ is dense in X^*, and that $f(X) = X^*$ if X is complete.

Solution. To prove density, let $[(x_n)_{n \in \mathbb{Z}_{>0}}] \in X^*$, and let $\varepsilon > 0$. Choose N such that if $m, n \geq N$ then $d(x_m, x_n) < \frac{\varepsilon}{2}$. Then

$$\Delta\big(f(x_N), [(x_n)_{n \in \mathbb{Z}_{>0}}]\big) = \lim_{n \to \infty} d(x_N, x_n),$$

which is at most $\frac{\varepsilon}{2}$ because $d(x_N, x_n) < \frac{\varepsilon}{2}$ for $n \geq N$. (The limit exists by Problem 3.23.) In particular, $\Delta\big(f(x_N), [(x_n)_{n \in \mathbb{Z}_{>0}}]\big) < \varepsilon$.

Now assume X is complete. Then $f(X)$ is complete, because f is isometric. Therefore it suffices to prove that a complete subset of a metric space is closed. (A subset of a metric space which is both closed and dense must be equal to the whole space.)

Accordingly, let Y be a metric space, and let $E \subset Y$ be a complete subset. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in E which converges to some point $y \in Y$; we show $y \in E$. (By a theorem proved in class, this is sufficient to verify that E is closed.) Now $(x_n)_{n \in \mathbb{Z}_{>0}}$ converges, and is therefore Cauchy. Since E is complete, there is $x \in E$ such that $x_n \to x$. By uniqueness of limits, we have $x = y$. Thus $y \in E$, as desired. □

Problem A. Prove the equivalence of four definitions of the lim sup of a sequence. That is, prove the following theorem.

Theorem. Let $(a_n)_{n \in \mathbb{Z}}$ be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of $(a_n)_{n \in \mathbb{Z}}$ in $[-\infty, \infty]$. Define numbers r, s, t, and $u \in [-\infty, \infty]$ as follows:

(1) $r = \sup(E)$.
(2) $s \in E$ and for every $x > s$, there is $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies $a_n < x$.
(3) $t = \inf_{n \in \mathbb{Z}_{>0}} \sup_{k \geq n} a_k$.
(4) $u = \lim_{n \to \infty} \sup_{k \geq n} a_k$.

Prove that s is uniquely determined by (2), that the limit in (4) exists in $[-\infty, \infty]$, and that $r = s = t = u$.

Note: You do not need to repeat the part that is done in the book (Theorem 3.17).

Solution. Theorem 3.17 of Rudin implies that s is uniquely determined by (2) and that $s = r$.

For $n \in \mathbb{Z}_{>0}$, define $b_n = \sup_{k \geq n} a_k$, which exists in $(-\infty, \infty]$. We clearly have

$$\{a_k : k \geq n+1\} \subset \{a_k : k \geq n\},$$

so that $\sup_{k \geq n+1} a_k \leq \sup_{k \geq n} a_k$. This shows that the sequence in (4), which has values in $(-\infty, \infty]$, is nonincreasing. Therefore it has a limit $u \in [-\infty, \infty]$, and moreover

$$u = \inf_{n \in \mathbb{Z}_{>0}} b_n = \inf_{n \in \mathbb{Z}_{>0}} \sup_{k \geq n} a_k = t.$$

We finish the proof by showing that $s \leq t$ and $t \leq r$.

To show that $s \leq t$, let $x > s$; we show that $x > t$. Choose y with $x > y > s$. By the definition of s, there is $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies $a_n < y$. This implies that $y \geq \sup_{n \geq N} a_n$, so that $x > \sup_{n \geq N} a_n$. It follows that x is not a lower bound for $\left\{\sup_{n \geq N} a_n : N \in \mathbb{Z}_{>0}\right\}$. So $x > t$ by the definition of a greatest lower bound.

To show that $t \geq r$, let $x > t$; we show that $x \geq r$. Since $x > t$, it follows that x is not a lower bound for the set $\left\{\sup_{n \geq N} a_n : N \in \mathbb{Z}_{>0}\right\}$. Accordingly, there is $N_0 \in \mathbb{Z}_{>0}$ such that $\sup_{n \geq N_0} a_n < x$. In particular, $n \geq N_0$ implies $a_n < x$. Now let $(a_{k(n)})_{n \in \mathbb{Z}}$ be any convergent subsequence of $(a_n)_{n \in \mathbb{Z}}$. Choose $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies $k(n) \geq N_0$. Then $n \geq N$ implies $a_{k(n)} < x$, from which it follows that $\lim_{n \to \infty} a_{k(n)} \leq x$. This shows that x is an upper bound for the set E, so that $x \geq \sup(E) = r$. $\square$