MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 8

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.1: Let $f \colon \mathbb{R} \to \mathbb{R}$ satisfy $\lim_{h \to 0} \big[f(x+h) - f(x-h) \big] = 0$ for all $x \in \mathbb{R}$. Is f necessarily continuous?

Solution. No. The simplest counterexample is

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Clearly

$$\lim_{h \to 0} \big[f(x+h) - f(x-h) \big] = \lim_{h \to 0} 0 = 0.$$

However, $f(0) \neq 0$, so f is not continuous at 0. For all other $x \in \mathbb{R}$, we also have

$$\lim_{h \to 0} \big[f(x+h) - f(x-h) \big] = \lim_{h \to 0} 0 = 0.$$

This completes the solution. □

More generally, let $f_0 \colon \mathbb{R} \to \mathbb{R}$ be continuous. Fix $x_0 \in \mathbb{R}$, and fix $y_0 \in \mathbb{R}$ with $y_0 \neq f_0(x_0)$. Then the function given by

$$f(x) = \begin{cases} f_0(x) & x \neq x_0 \\ y_0 & x = x_0 \end{cases}$$

is a counterexample. There are even examples with a nonremovable discontinuity, such as

$$f(x) = \begin{cases} \frac{1}{|x|} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Problem 4.3: Let X be a metric space, and let $f \colon X \to \mathbb{R}$ be continuous. Let $Z(f) = \{ x \in X \colon f(x) = 0 \}$. Prove that $Z(f)$ is closed.

Solution 1. The set $Z(f)$ is equal to $f^{-1}(\{0\})$. Since $\{0\}$ is a closed subset of $\mathbb{R}$ and f is continuous, it follows from the Corollary to Theorem 4.8 of Rudin that $Z(f)$ is closed in X. □

Solution 2. We show that $X \setminus Z(f)$ is open. Let $x \in X \setminus Z(f)$. Then $f(x) \neq 0$. Set $\varepsilon = \frac{1}{2} |f(x)| > 0$. Choose $\delta > 0$ such that $y \in X$ and $d(x, y) < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Then $f(y) \neq 0$ for $y \in N_\delta(x)$. Thus $N_\delta(x) \subset X \setminus Z(f)$ with $\delta > 0$. This shows that $X \setminus Z(f)$ is open. □

Note that we could have taken $\varepsilon = |f(x)|$. Also, there is no need to do anything special if $Z(f)$ is empty, or even to mention the that case separately: the argument works (vacuously) just as well in that case.

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Solution 3 (sketch). We show $Z(f)$ contains all its limit points. Let x be a limit point of $Z(f)$. Then there is a sequence $(x_n)_{n\in\mathbb{Z}}$ in $Z(f)$ such that $x_n \to x$. Since f is continuous and $f(x_n) = 0$ for all n, we have

$$f(x) = \lim_{n\to\infty} f(x_n) = 0.$$

So $x \in Z(f)$. ☐

Again, there is no need to treat separately the case in which $Z(f)$ has no limit points.

Problem 4.4: Let X and Y be metric spaces, and let $f, g\colon X \to Y$ be continuous functions. Let $E \subset X$ be dense. Prove that $f(E)$ is dense in $f(X)$. Prove that if $f(x) = g(x)$ for all $x \in E$, then $f = g$.

Solution. We first show that $f(E)$ is dense in $f(X)$. Let $y \in f(X)$. Choose $x \in X$ such that $f(x) = y$. Since E is dense in X, there is a sequence $(x_n)_{n\in\mathbb{Z}}$ in E such that $x_n \to x$. Since f is continuous, it follows that $f(x_n) \to f(x)$. Since $f(x_n) \in f(E)$ for all n, this shows that $x \in \overline{f(E)}$.

Now assume that $f(x) = g(x)$ for all $x \in E$; we prove that $f = g$. It suffices to prove that $F = \{x \in X \colon f(x) = g(x)\}$ is closed in X, and we prove this by showing that $X \setminus F$ is open. Thus, let $x_0 \in F$. Set $\varepsilon = \frac{1}{2} d(f(x_0),\, g(x_0)) > 0$. Choose $\delta_1 > 0$ such that if $x \in X$ satisfies $d(x, x_0) < \delta$, then $d(f(x),\, f(x_0)) < \varepsilon$. Choose $\delta_2 > 0$ such that if $x \in X$ satisfies $d(x, x_0) < \delta$, then $d(g(x),\, g(x_0)) < \varepsilon$. Set $\delta = \min(\delta_1, \delta_2)$. If $d(x, x_0) < \delta$, then (using the triangle inequality several times)

$$\begin{aligned} d(f(x),\, g(x)) &\geq d(f(x_0),\, g(x_0)) - d(f(x),\, f(x_0)) - d(g(x),\, g(x_0)) \\ &> d(f(x_0),\, g(x_0)) - \varepsilon - \varepsilon = 0. \end{aligned}$$

So $f(x) \neq g(x)$. This shows that $N_\delta(x_0) \subset X \setminus F$, so that $X \setminus F$ is open. ☐

The second part is closely related to Problem 4.3. If $Y = \mathbb{R}$ (or $\mathbb{C}^n$, or $\ldots$), then $\{x \in X \colon f(x) = g(x)\} = Z(f - g)$, and $f - g$ is continuous when f and g are. For general Y, however, this solution fails, since $f - g$ isn't defined. The argument given is the analog of Solution 2 to Problem 4.3. The analog of Solution 3 to Problem 4.3 also works the same way: F is closed because if $x_n \to x$ and $f(x_n) = g(x_n)_{n\in\mathbb{Z}}$ for all n, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n)_{n\in\mathbb{Z}}$. The analog of Solution 1 can actually be made to work in the following way (using the fact that the product of two metric spaces is again a metric space): Define $h\colon X \to Y \times Y$ by $h(x) = (f(x),\, g(x))$. Then h is continuous and $D = \{(y, y)\colon y \in Y\} \subset Y \times Y$ is closed, so $\{x \in X \colon f(x) = g(x)\} = h^{-1}(D)$ is closed.

Problem 4.6: Let $E \subset \mathbb{R}$ be compact, and let $f\colon E \to \mathbb{R}$ be a function. Prove that f is continuous if and only if the graph $G(f) = \{(x,\, f(x))\colon x \in E\} \subset \mathbb{R}^2$ is compact.

Remark: This statement is my interpretation of what was intended. Normally one would assume that E is supposed to be a compact subset of an arbitrary metric space X, and that f is supposed to be a function from E to some other metric space Y. (In fact, one might as well assume $E = X$.) The proofs are all the same (with one exception, noted below), but require the notion of the product of two metric spaces. We make $X \times Y$ into a metric space via the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2};$$

there are other choices which are easier to deal with and work just as well.

We give several solutions for each direction. We first show that if f is continuous then $G(f)$ is compact.

Solution 1 (sketch). The map $x \mapsto (x, f(x))$ is easily checked to be continuous, and $G(f)$ is the image of the compact set E under this map, so $G(f)$ is compact by Theorem 4.14 of Rudin. □

Solution 2 (sketch). The graph of a continuous function is closed, as can be verified by arguments similar to those of Solutions 2 and 3 to Problem 4.3. The graph is a subset of $E \times f(E)$. This set is bounded (clear) and closed (check this!) in $\mathbb{R}^2$, and is therefore compact. (Note: This does not work for general metric spaces. However, it is true in general that the product of two compact sets, with the product metric, is compact.) Therefore the closed subset $G(f)$ is compact. □

Now we show that if $G(f)$ is compact then f is continuous.

Solution 1. We know that the function $g_0 \colon E \times \mathbb{R} \to E$, given by $g_0(x, y) = x$, is continuous. (See Example 4.11 of Rudin.) Therefore $g = g_0|_{G(f)} \colon G(f) \to E$ is continuous. Also g is bijective (because f is a function). Since $G(f)$ is compact, it follows (Theorem 4.17 of Rudin) that $g^{-1} \colon E \to G(f)$ is continuous. Furthermore, the function $h \colon E \times \mathbb{R} \to \mathbb{R}$, given by $h(x, y) = y$, is continuous, again by Example 4.11 of Rudin. Therefore $f = h \circ g^{-1}$ is continuous. □

Solution 2 (sketch). Let $(x_n)_{n \in \mathbb{Z}}$ be a sequence in E with $x_n \to x$. We will show that $f(x_n) \to f(x)$.

We claim that if Y is any metric space, $(y_n)_{n \in \mathbb{Z}}$ is a sequence in Y, $y \in Y$, and every subsequence of $(y_n)_{n \in \mathbb{Z}}$ has in turn a subsubsequence which converges to y, then $y_n \to y$. To prove the claim, suppose that $(y_n)_{n \in \mathbb{Z}}$ does not converge to y. Then there is a subsequence $(y_{k(n)})_{n \in \mathbb{Z}}$ of $(y_n)_{n \in \mathbb{Z}}$ such that $\inf_{n \in \mathbb{Z}_{>0}} d(y_{k(n)}, y) > 0$. No subsequence of $(y_{k(n)})_{n \in \mathbb{Z}}$ can converge to y.. The claim is proved.

We now claim that every subsequence of $(f(x_n)_{n \in \mathbb{Z}})$ has in turn a subsubsequence which converges to $f(x)$.

To prove the claim, let $n \mapsto k(n)$ be a strictly increasing function from $\mathbb{Z}_{>0}$ to $\mathbb{Z}_{>0}$. We will prove that $(f(x_{k(n)})_{n \in \mathbb{Z}})$ has a subsequence which converges to $f(x)$. If $(x_{k(n)})_{n \in \mathbb{Z}}$ is eventually constant, then already $f(x_{k(n)}) \to f(x)$. Otherwise, since $x_{k(n)} \to x$, one can check that $\{x_{k(n)} \colon n \in \mathbb{Z}_{>0}\}$ is an infinite set. Therefore so is $\big\{(x_{k(n)}, f(x_{k(n)})) \colon n \in \mathbb{Z}_{>0}\big\} \subset G(f)$. Since $G(f)$ is compact, this set has a limit point, say (a, b). It is easy to check that a must equal x. Since $G(f)$ is compact, it is closed, so $b = f(x)$. Since (a, b) is a limit point of $G(f)$, there is a subsequence of $\big((x_{k(n)}, f(x_{k(n)}))\big)_{n \in \mathbb{Z}}$ which converges to (a, b). Using continuity of projection onto the second coordinate, we get a subsequence of $\big(f(x_{k(n)})\big)_{n \in \mathbb{Z}}$ which converges to $f(x)$. The claim is proved.

By the previous claim, the proof is complete. □

Solution 3 (sketch). We first claim that the range $Y = \{f(x) \colon x \in E\}$ of f is compact. Indeed, Y is the image of $G(f)$ under the map $(x, y) \mapsto y$, which is continuous by Example 4.11 of Rudin. So Y is compact by Theorem 4.14 of Rudin. The claim is proved.

We claim that if $x_0 \in E$ and $V \subset Y$ is an open set containing $f(x_0)$, then there is an open set $U \subset E$ containing x_0 such that $f(U) \subset V$. We prove the claim. For each $y \in Y \setminus V$, the point (x_0, y) is not in the closed set $G \subset E \times \mathbb{R}$. Therefore there exist open sets $R_y \subset E$ containing x_0 and $S_y \subset Y$ containing y such that $(R_y \times S_y) \cap G = \varnothing$. Since $Y \setminus V$ is compact, there are n and $y(1), y(2), \ldots, y(n) \in Y \setminus V$ such that the sets $S_{y(1)}, S_{y(2)}, \ldots, S_{y(n)}$ cover $Y \setminus V$. Set $U = R_{y(1)} \cap R_{y(2)} \cap \cdots \cap R_{y(n)}$. Then $f(U) \subset V$. The claim is proved.

Continuity of f follows from the last claim. (Why?) $\square$

Problem 4.7: Define $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Prove:

(1) f is bounded.
(2) f is not continuous at $(0,0)$.
(3) The restriction of f to every straight line in $\mathbb{R}^2$ is continuous.
(4) g is not bounded on any neighborhood of $(0,0)$.
(5) g is not continuous at $(0,0)$.
(6) The restriction of g to every straight line in $\mathbb{R}^2$ is continuous.

Solution (sketch). To prove (1), use the inequality $2ab \leq a^2 + b^2$ (which follows from $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$). Taking $a = |x|$ and $b = y^2$, we get $2|x|y^2 \leq x^2 + y^4$, which implies $|f(x,y)| \leq \frac{1}{2}$ for all $(x,y) \in \mathbb{R}^2$.

To prove (2) set $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$. Then $(x_n, y_n) \to (0,0)$, but $f(x_n, y_n) = \frac{1}{2} \not\to 0 = f(0,0)$.

We prove (3). Clearly f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$, so the restriction of f to every straight line in $\mathbb{R}^2$ not going through $(0,0)$ is clearly continuous. Furthermore, the restriction of f to the y-axis is given by $(0,y) \mapsto 0$, which is clearly continuous.

Every other line has the form $y = ax$ for some $a \in \mathbb{R}$. We have

$$f(x, ax) = \frac{a^2 x}{1 + a^4 x^2}$$

for all $x \in \mathbb{R}$, so the restriction of f to this line is given by the continuous function

$$(x,y) \mapsto \frac{a^2 x}{1 + a^4 x^2}.$$

For (4), set $x_n = \frac{1}{n^3}$ and $y_n = \frac{1}{n}$. Then $(x_n, y_n) \to (0,0)$, but $g(x_n, y_n) = n \to \infty$.

Part (5) is immediate from (4).

We prove (6). Clearly g is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$, so the restriction of g to every straight line in $\mathbb{R}^2$ not going through $(0,0)$ is clearly continuous. Furthermore, the restriction of g to the y-axis is given by $(0,y) \mapsto 0$, which is clearly continuous.

Every other line has the form $y = ax$ for some $a \in \mathbb{R}$. We have

$$g(x, ax) = \frac{a^3 x}{1 + a^6 x^4}$$

for all $x \in \mathbb{R}$, so the restriction of g to this line is given by the function

$$(x, y) \mapsto \frac{a^3 x}{1 + a^6 x^4},$$

which is clearly continuous. □

Problem 4.8: Let $E \subset \mathbb{R}$ be bounded, and let $f \colon E \to \mathbb{R}$ be uniformly continuous. Prove that f is bounded. Show that a uniformly continuous function on an unbounded subset of $\mathbb{R}$ need not be bounded.

Solution. Choose $\delta > 0$ such that if $x, y \in E$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Choose $a, b \in \mathbb{R}$ such that $E \subset [a, b]$. Choose $n \in \mathbb{Z}_{>0}$ such that $n\delta > b - a$. For $k = 0, 1, \ldots, n$ define $t_k = a + \frac{k}{n}(b - a)$. Let S be the finite set of those $k \in \{0, 1, \ldots, n\}$ for which $E \cap [t_{k-1}, t_k] \neq \varnothing$. Thus $E \subset \bigcup_{k \in S} [t_{k-1}, t_k]$. For $k \in S$ choose $c_k \in E \cap [t_{k-1}, t_k]$. Set $M = 1 + \max_{k \in S} |f(c_k)|$.

We show that $|f(x)| \leq M$ for all $x \in E$. For such x, choose $k \in S$ such that $x \in [t_{k-1}, t_k]$. Then $|x - c_k| \leq \frac{1}{n}(b - a) < \delta$, so $|f(x) - f(c_k)| < 1$. Thus $|f(x)| \leq |f(x) - f(c_k)| + |f(c_k)| < 1 + M$.

As a counterexample with E unbounded, take $E = \mathbb{R}$ and $f(x) = x$ for all x. □

Problem 4.9: Let X and Y be metric spaces, and let $f \colon X \to Y$ be a function. Prove that f is uniformly continuous if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $E \subset X$ satisfies $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$.

Solution. Let f be uniformly continuous, and let $\varepsilon > 0$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \frac{1}{2}\varepsilon$. Let $E \subset X$ satisfy $\operatorname{diam}(E) < \delta$. We show that $\operatorname{diam}(f(E)) < \varepsilon$. Let $y_1, y_2 \in f(E)$. Choose $x_1, x_2 \in E$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $d(x_1, x_2) < \delta$, so $d(f(x_1), f(x_2)) < \frac{1}{2}\varepsilon$. This shows that $d(y_1, y_2) < \frac{1}{2}\varepsilon$ for all $y_1, y_2 \in f(E)$. Therefore

$$\operatorname{diam}(f(E)) = \sup_{y_1, y_2 \in E} d(y_1, y_2) \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

Now assume that for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $E \subset X$ satisfies $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$. We prove that f is uniformly continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ as in the hypotheses. Let $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$. Set $E = \{x_1, x_2\}$. Then $\operatorname{diam}(E) < \delta$. So $d(f(x_1), f(x_2)) = \operatorname{diam}(f(E)) < \varepsilon$. □