MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 9

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.10: Use the fact that infinite subsets of compact sets have limit points to give an alternate proof that if X and Z are metric spaces with X compact, and $f \colon X \to Z$ is continuous, then f is uniformly continuous.

Solution. Assume that f is not uniformly continuous. Choose $\varepsilon > 0$ for which the definition of uniform continuity fails. Then for every $n \in \mathbb{Z}_{>0}$ there are $x_n, \, y_n \in X$ such that $d(x_n, y_n) < \frac{1}{n}$ and $d\big(f(x_n), \, f(y_n)\big) \geq \varepsilon$. Since X is compact, by Theorem 3.6(a) of Rudin the sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ has a convergent subsequence, say $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$. Let $x = \lim_{n \to \infty} x_{k(n)}$. Since $d\big(x_{k(n)}, \, y_{k(n)}\big) < \frac{1}{k(n)} \leq \frac{1}{n}$, we also have $\lim_{n \to \infty} y_{k(n)} = x$.

We claim that f is discontinuous at x. This will complete the proof. To prove the claim, suppose that f is continuous at x. Then

$$\lim_{n \to \infty} f(x_{k(n)}) = \lim_{n \to \infty} f(y_{k(n)}) = f(x).$$

Choose $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies

$$d\big(f(x_{k(n)}), \, f(x)\big) < \frac{\varepsilon}{3} \qquad \text{and} \qquad d\big(f(y_{k(n)}), \, f(x)\big) < \frac{\varepsilon}{3}.$$

Then

$$d\big(f(x_{k(N)}), \, f(y_{k(N)}\big) \leq d\big(f(x_{k(n)}), \, f(x)\big) + d\big(f(x), \, f(y_{k(n)})\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3},$$

but by construction $d\big(f(x_{k(N)}), \, f(y_{k(N)}\big) \geq \varepsilon$. This contradiction proves the claim. ∎

Remark: It is a serious mistake to simply claim that the sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ has a convergent subsequence $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ and the sequence $(y_n)_{n \in \mathbb{Z}_{>0}}$ has a convergent subsequence $(y_{k(n)})_{n \in \mathbb{Z}_{>0}}$. If one chooses convergent subsequences of $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$, they must be called, say, $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ and $(y_{l(n)})_{n \in \mathbb{Z}_{>0}}$ for *different* functions $k, \, l \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

It is nevertheless possible to carry out a proof by passing to convergent subsequences of $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$. The following solution shows how it can be done. This solution is not recommended here, but in other situations it may be the only way to proceed.

Alternate solution. Assume that f is not uniformly continuous. Choose $\varepsilon > 0$ for which the definition of uniform continuity fails. Then for every $n \in \mathbb{Z}_{>0}$ there are $x_n, \, y_n \in X$ such that $d(x_n, y_n) < \frac{1}{n}$ and $d\big(f(x_n), \, f(y_n)\big) \geq \varepsilon$. Since X is compact,

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by Theorem 3.6(a) of Rudin the sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ has a convergent subsequence, say $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$. Then $(y_{k(n)})_{n \in \mathbb{Z}_{>0}}$ is a sequence in a compact metric space, and therefore, again by Theorem 3.6(a) of Rudin, it has a convergent subsequence, say $(y_{k(r(n))})_{n \in \mathbb{Z}_{>0}}$. Let $l = k \circ r$. Then $(x_{l(n)})_{n \in \mathbb{Z}_{>0}}$ is a subsequence of the convergent sequence $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$, and therefore converges.

Let $x = \lim_{n \to \infty} x_{l(n)}$ and $y = \lim_{n \to \infty} y_{l(n)}$. We claim that $x = y$. To prove the claim, let $\varepsilon > 0$; we show that $d(x, y) < \varepsilon$. Choose $N_1, N_2, N_3 \in \mathbb{Z}_{>0}$ so large that $n \geq N_1$ implies $d(x_{l(n)}, x) < \frac{\varepsilon}{3}$, so large that $n \geq N_2$ implies $d(y_{l(n)}, y) < \frac{\varepsilon}{3}$, and so large that $n \geq N_3$ implies $\frac{1}{n} < \frac{\varepsilon}{3}$. Set $n = \max(N_1, N_2, N_3)$. Then

$$d(x,y) \leq d(x, x_{l(n)}) + d(x_{l(n)}, y_{l(n)}) + d(y_{l(n)}, y) < \frac{\varepsilon}{3} + \frac{1}{l(n)} + \frac{\varepsilon}{3} \leq \frac{\varepsilon}{3} + \frac{1}{n} + \frac{\varepsilon}{3} < \varepsilon.$$

The claim is proved.

We now know that $\lim_{n \to \infty} y_{l(n)} = x = \lim_{n \to \infty} x_{l(n)}$.

We claim that f is discontinuous at x. This will complete the proof. To prove the claim, suppose that f is continuous at x. Then

$$\lim_{n \to \infty} f(x_{l(n)}) = \lim_{n \to \infty} f(y_{l(n)}) = f(x).$$

Choose $N \in \mathbb{Z}_{>0}$ such that $n \geq N$ implies

$$d\big(f(x_{l(n)}), f(x)\big) < \frac{\varepsilon}{3} \qquad \text{and} \qquad d\big(f(y_{l(n)}), f(x)\big) < \frac{\varepsilon}{3}.$$

Then

$$d\big(f(x_{l(N)}), f(y_{l(N)})\big) \leq d\big(f(x_{l(n)}), f(x)\big) + d\big(f(x), f(y_{l(n)})\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3},$$

but by construction $d\big(f(x_{l(N)}), f(y_{l(N)})\big) \geq \varepsilon$. This contradiction proves the claim. □

Problem 4.11: Let X and Y be metric spaces.

(1) Let $f \colon X \to Y$ be uniformly continuous. Prove that if $(x_n)_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in X, then $(f(x_n))_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in Y.

(2) Use Part (1) to prove that if Y is complete, $E \subset X$ is dense, and $f_0 \colon E \to Y$ is uniformly continuous, then there is a unique continuous function $f \colon X \to Y$ such that $f|_E = f_0$.

Solution to (1). We prove the first statement. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in X. Let $\varepsilon > 0$. Choose $\delta > 0$ such that if $s_1, s_2 \in X$ satisfy $d(s_1, s_2) < \delta$, then $d(f(s_1), f(s_2)) < \varepsilon$. Choose $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ satisfy $m, n \geq N$, then $d(x_m, x_n) < \delta$. Then whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m, n \geq N$, we have $d(x_m, x_n) < \delta$, so that $d(f(x_m), f(x_n)) < \varepsilon$. This shows that $(f(x_n))_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence. □

For (2), the neatest arrangement I can think of is to prove the following lemmas first.

Lemma 1. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f \colon E \to Y$ be uniformly continuous. Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in E which converges to some point in X. Then $\lim_{n \to \infty} f(x_n)$ exists in Y.

Proof. We know that convergent sequences are Cauchy. This is therefore immediate from the first part of the problem and the definition of completeness. □

Lemma 2. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f \colon E \to Y$ be uniformly continuous. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $w, x \in X$ satisfy $d(w, x) < \delta$, and whenever $(r_n)_{n \in \mathbb{Z}_{>0}}$ and $(s_n)_{n \in \mathbb{Z}_{>0}}$ are sequences in E such that $r_n \to w$ and $s_n \to x$, then

$$d\left(\lim_{n \to \infty} f(r_n), \, \lim_{n \to \infty} f(s_n) \right) < \varepsilon.$$

The limits exist by Lemma 1.

Proof of Lemma 2. Let $\varepsilon > 0$. Choose $\rho > 0$ such that whenever $w, \, x \in E$ satisfy $d(w, x) < \rho$, then $d(f(w), \, f(x)) < \frac{\varepsilon}{2}$. Set $\delta = \frac{\rho}{2} > 0$. Let $w, \, x \in X$ satisfy $d(w, x) < \delta$, and let $(r_n)_{n \in \mathbb{Z}_{>0}}$ and $(s_n)_{n \in \mathbb{Z}_{>0}}$ be sequences in E such that $r_n \to w$ and $s_n \to x$. Let $y = \lim_{n \to \infty} f(r_n)$ and $z = \lim_{n \to \infty} f(s_n)$. (These exist by Lemma 1.) Choose N so large that for all $n \in \mathbb{Z}_{>0}$ with $n \geq N$, the following four conditions are all satisfied:

- $d(r_n, w) < \frac{\rho}{4}$.
- $d(s_n, x) < \frac{\rho}{4}$.
- $d(f(r_n), \, y) < \frac{\varepsilon}{4}$.
- $d(f(s_n), \, z) < \frac{\varepsilon}{4}$.

We then have

$$d(r_N, s_N) \leq d(r_N, w) + d(w, x) + d(x, s_N) < \frac{\rho}{4} + \frac{\rho}{2} + \frac{\rho}{4} = \rho.$$

Therefore $d(f(r_N), \, f(s_N)) < \frac{\varepsilon}{2}$. So

$$d(y, z) \leq d(y, \, f(r_N)) + d(f(r_N), \, f(s_N)) + d(f(s_N), \, z) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon,$$

as desired. □

Theorem 3. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f \colon E \to Y$ be uniformly continuous. Then there is a unique continuous function $f \colon X \to Y$ such that $f|_E = f_0$.

Proof. If f exists, then it is unique by Problem 4.4, which was in the previous assignment. So we prove existence. For $x \in X$, we want to define $f(x)$ by choosing a sequence $(r_n)_{n \in \mathbb{Z}_{>0}}$ in E with $\lim_{n \to \infty} r_n = x$ and then setting $f(x) = \lim_{n \to \infty} f_0(r_n)_{n \in \mathbb{Z}_{>0}}$. We know that such a sequence exists because E is dense in X. We know that $\lim_{n \to \infty} f_0(r_n)$ exists, by Lemma 1. However, we must show that $\lim_{n \to \infty} f_0(r_n)$ only depends on x, not on the sequence $(r_n)_{n \in \mathbb{Z}_{>0}}$.

To prove this, let $(r_n)_{n \in \mathbb{Z}_{>0}}$ and $(s_n)_{n \in \mathbb{Z}_{>0}}$ be sequences in E with

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} s_n = x.$$

We claim that $\lim_{n \to \infty} f_0(r_n) = \lim_{n \to \infty} f_0(s_n)$. To prove the claim, let $\varepsilon > 0$; we show that

$$d\left(\lim_{n \to \infty} f_0(r_n), \, \lim_{n \to \infty} f_0(s_n) \right) < \varepsilon.$$

Choose $\delta > 0$ according to Lemma 2. We certainly have $d(x, x) < \delta$. Therefore the conclusion of Lemma 2 gives

$$d\left(\lim_{n \to \infty} f_0(r_n), \, \lim_{n \to \infty} f_0(s_n) \right) < \varepsilon,$$

proving the claim.

We now get a well defined function $f\colon X \to Y$ by, for $x \in X$, choosing any sequence $(r_n)_{n\in\mathbb{Z}_{>0}}$ in E with $\lim_{n\to\infty} r_n = x$ and defining $f(x) = \lim_{n\to\infty} f_0(r_n)$. By considering the constant sequence $x_n = x$ for all n, we see immediately that $f(x) = f_0(x)$ for $x \in E$.

We claim that f is continuous, in fact uniformly continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ according to Lemma 2. For $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, choose (by density of E, as above) sequences $(r_n)_{n\in\mathbb{Z}_{>0}}$ and $(s_n)_{n\in\mathbb{Z}_{>0}}$ in E such that $r_n \to x_1$ and $s_n \to x_2$. Then

$$d\left(\lim_{n\to\infty} f_0(r_n),\ \lim_{n\to\infty} f_0(s_n)\right) < \varepsilon.$$

By construction, we have $f(x_1) = \lim_{n\to\infty} f_0(r_n)$ and $f(x_2) = \lim_{n\to\infty} f_0(s_n)$. Therefore we have shown that $d(f(x_1),\ f(x_2)) < \varepsilon$, as desired. The claim is proved. $\square$

The point of stating Lemma 2 separately is that the proof that f is well defined, and the proof that f is continuous, use essentially the same argument. By putting that argument in a lemma, we avoid repeating it.

Problem 4.12: State precisely and prove the following: "A uniformly continuous function of a uniformly continuous function is uniformly continuous."

Here is the precise statement.

Proposition 4. Let X, Y, and Z be metric spaces. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be uniformly continuous functions. Then $g \circ f$ is uniformly continuous.

Solution. Let $\varepsilon > 0$. Choose $\rho > 0$ such that if $y_1, y_2 \in Y$ satisfy $d(y_1, y_2) < \rho$, then $d(g(y_1),\ g(y_2)) < \varepsilon$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1),\ f(x_2)) < \rho$. Then whenever $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, we have $d(f(x_1),\ f(x_2)) < \rho$, so that $d(g(f(x_1)),\ g(f(x_2))) < \varepsilon$. $\square$

Problem 4.14: Let $f\colon [0,1] \to [0,1]$ be continuous. Prove that there is $x \in [0,1]$ such that $f(x) = x$.

Solution. Define $g\colon [0,1] \to \mathbb{R}$ by $g(x) = x - f(x)$. Then g is continuous. Since $f(0) \in [0,1]$, we have $g(0) = -f(0) \leq 0$, while since $g(1) \in [0,1]$, we have $g(1) = 1 - f(1) \geq 0$. If $g(0) = 0$ then $x = 0$ satisfies the conclusion, while if $g(1) = 0$ then $x = 1$ satisfies the conclusion. Otherwise, $g(0) < 0$ and $g(1) > 0$, so Theorem 4.23 of Rudin provides $x \in (0,1)$ such that $g(x) = 0$. This x satisfies $f(x) = x$. $\square$

Something much more general is true, namely the Brouwer Fixed Point Theorem.

Theorem 5. Let $n \geq 1$, and let $B = \{x \in \mathbb{R}^n \colon \|x\| \leq 1\}$. Let $f\colon B \to B$ be continuous. Then there is $x \in B$ such that $f(x) = x$.

The proof requires higher orders of connectedness, and is best done with algebraic topology.

Problem 4.15: Prove that every continuous open map $f\colon \mathbb{R} \to \mathbb{R}$ is monotone.

Two solutions are presented,. the second of which is sketchy. The second is what I expect people to have done. The first is essentially a careful rearrangement of the ideas of the second, done so as to minimize the number of cases. (You will see when reading the second solution why this is desirable.)

Solution. It is convenient to break the solution into several lemmas.

Lemma 6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and open. Let $a, b \in \mathbb{R}$ satisfy $a < b$. Then $f(a) \neq f(b)$.

Proof. Since f is continuous and $[a, b]$ is compact, there are $c_1, c_2 \in [a, b]$ such that $m_1 = f(c_1)$ is the minimum value of f on $[a, b]$ and $m_2 = f(c_2)$ is the maximum value of f on $[a, b]$.

We claim that $c_1 \in \{a, b\}$. Suppose not. Then $m_1 \in f\big((a, b)\big)$, but $(-\infty, m_1) \cap f\big([a, b]\big) = \varnothing$. Therefore m_1 is not an interior point of $f\big((a, b)\big)$. This contradicts the assumption that f is an open map, and proves the claim.

Applying this argument to $-f$, we see that $c_2 \in \{a, b\}$.

Now, if $f(a) = f(b)$, then $m_1 = m_2$. So $f\big((a, b)\big) = \{m_1\}$, which is not an open set. This contradiction proves the lemma. □

Lemma 7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and open. Let $a, b \in \mathbb{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbb{R}$ satisfies $x < a$, we have $f(x) < f(a)$.

Proof. We can't have $f(x) = f(a)$, by Lemma 6. If $f(x) = f(b)$, we again have a contradiction by Lemma 6. If $f(x) > f(b)$, then the Intermediate Value Theorem provides $z \in (x, a)$ such that $f(z) = f(b)$. Since $z < b$, this contradicts Lemma 6. If $f(a) < f(x) < f(b)$, then the Intermediate Value Theorem provides $z \in (a, b)$ such that $f(z) = f(x)$. Since $x < z$, this again contradicts Lemma 6. The only remaining possibility is $f(x) < f(a)$. □

Lemma 8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and open. Let $a, b \in \mathbb{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbb{R}$ satisfies $b < x$, we have $f(b) < f(x)$.

Proof. Apply Lemma 7 to the function $x \mapsto -f(-x)$. □

Lemma 9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and open. Let $a, b \in \mathbb{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbb{R}$ satisfies $a < x < b$, we have $f(a) < f(x) < f(b)$.

Proof. Lemma 6 implies that $f(x)$ is equal to neither $f(a)$ nor $f(b)$. If $f(x) < f(a)$, we apply Lemma 8 to $-f$, with b and x interchanged, to get $f(b) < f(x)$. This implies $f(b) < f(a)$, which contradicts the hypotheses. If $f(x) > f(b)$, we apply Lemma 7 to $-f$, with a and x interchanged, to get $f(a) > f(x)$. This again contradicts the hypotheses. □

Corollary 10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and open. Let $a, b \in \mathbb{R}$ satisfy $a < b$, and suppose that $f(a) < f(b)$. Let $x \in \mathbb{R}$. Then:

(1) If $x \leq a$ then $f(x) \leq f(a)$.
(2) If $x \leq b$ then $f(x) \leq f(b)$.
(3) If $x \geq a$ then $f(x) \geq f(a)$.
(4) If $x \geq b$ then $f(x) \geq f(b)$.

Proof. In any of the four cases, if we have equality ($x = a$ or $x = b$), the conclusion is obvious. With strict inequality, Part (1) follows from Lemma 7, and Part (4) follows from Lemma 8. Part (2) follows from Lemma 9 if $x > a$, from Lemma 7 if $x < a$, and is trivial if $x = a$. Part (3) follows from Lemma 9 if $x < b$, from Lemma 8 if $x > b$, and is trivial if $x = b$. □

I won't actually use Part (2); it is included for symmetry.

Now we prove the result. Choose arbitrary $c, d \in \mathbb{R}$ with $c < d$. We have $f(c) \neq f(d)$ by Lemma 6. Suppose first that $f(c) < f(d)$. Let $r, s \in \mathbb{R}$ satisfy $r \leq s$. We prove that $f(r) \leq f(s)$, and there are several cases. I will try to arrange this to keep the number of cases as small as possible.

Case 1: $r \leq c \leq s$. Then $f(r) \leq f(c) \leq f(s)$ by Parts (1) and (3) of Corollary **??**, taking $a = c$ and $b = d$.

Case 2: $r \leq s \leq a$. Then $f(s) \leq f(a)$ by Part (1) of Corollary **??**, taking $a = c$ and $b = d$. Further, $f(r) \leq f(s)$ by Part (1) of Corollary **??**, taking $a = s$ and $b = d$.

Case 3: $a \leq r \leq s$. Then $f(a) \leq f(r)$ by Part (3) of Corollary **??**, taking $a = c$ and $b = d$. Further, $f(r) \leq f(s)$ by Part (4) of Corollary **??**, taking $a = c$ and $b = r$.

The case $f(c) > f(d)$ follows by applying the preceding argument to $-f$. □

Alternate solution (brief sketch). Suppose f is not monotone; we prove that f is not open. Since f isn't nondecreasing, there exist $a, b \in \mathbb{R}$ such that $a < b$ and $f(a) > f(b)$; and since f isn't nonincreasing, there exist $c, d \in \mathbb{R}$ such that $c < d$ and $f(c) < f(d)$. Now there are various cases depending on how a, b, c, and d are arranged in $\mathbb{R}$, and depending on how $f(a)$ and $f(b)$ relate to $f(c)$ and $f(d)$. There are 13 possible ways for a, b, c, and d to be arranged in $\mathbb{R}$, namely:

$$a < b < c < d \tag{1}$$

$$a < b = c < d \tag{2}$$

$$a < c < b < d \tag{3}$$

$$a < c < b = d \tag{4}$$

$$a < c < d < b \tag{5}$$

$$a = c < b < d \tag{6}$$

$$a = c < b = d \tag{7}$$

$$a = c < d < b \tag{8}$$

$$c < a < b < d \tag{9}$$

$$c < a < b = d \tag{10}$$

$$c < a < d < b \tag{11}$$

$$c < a = d < b \tag{12}$$

$$c < d < a < b. \tag{13}$$

Of these, the arrangement (7) gives an immediate contradiction. For each of the others, we find $x < y < z$ such that $f(y) < f(x), f(z)$ (so that f is not open by Lemma 7 of the previous solution), or such that $f(y) > f(x), f(z)$ (so that f is not open by Lemma 8 of the previous solution). Many cases break down into subcases depending on how the values of f are arranged. We illustrate by treating the arrangement (1).

Suppose $a < b < c < d$ and $f(b) < f(c)$. Set

$$x = a, \qquad y = b, \qquad \text{and} \qquad z = d.$$

Then $x < y < z$ and $f(y) < f(x),\ f(z)$, so Lemma 7 applies. Suppose, on the other hand, that $a < b < c < d$ and $f(b) \geq f(c)$. Set

$$x = a, \qquad y = c, \qquad \text{and} \qquad z = d.$$

Then again $x < y < z$ and $f(y) < f(x),\ f(z)$, so Lemma 7 applies. □

Problem 4.16: For $x \in \mathbb{R}$, define $[x]$ by the relations $[x] \in \mathbb{Z}$ and $x - 1 < [x] \leq x$ (this is called the "integer part of x" or the "greatest integer function"), and define $(x) = x - [x]$ (this is called the "fractional part of x", but the notation (x) is not standard). What discontinuities do the functions $x \mapsto [x]$ and $x \mapsto (x)$ have?

Solution (sketch). Both functions are continuous at all noninteger points, since $x \in (n,\, n+1)$ implies $[x] = n$ and $(x) = x - n$; both expressions are continuous on the interval $(n,\, n+1)$.

Both functions have jump discontinuities at all integers: for $n \in \mathbb{Z}$, we have

$$\lim_{x \to n^+} [x] = \lim_{x \to n^+} n = n = f(n) \qquad \text{and} \qquad \lim_{x \to n^-} [x] = \lim_{x \to n^-} (n-1) = n-1 \neq f(n),$$

and also

$$\lim_{x \to n^+} (x) = \lim_{x \to n^+} (x - n) = 0 = f(n)$$

and

$$\lim_{x \to n^-} (x) = \lim_{x \to n^-} [x - (n-1)] = 1 \neq f(n).$$

This completes the sketch of the solution. □

A solution must include proofs that both functions are continuous on $\mathbb{R} \setminus \mathbb{Z}$ and that both functions have jump discontinuities at every point of $\mathbb{Z}$. Merely stating these facts gets little credit.

Problem 4.18: Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

(By definition, we require $q > 0$. If $x = 0$ we take $p = 0$ and $q = 1$.) Prove that f is continuous at each point $x \in \mathbb{R} \setminus \mathbb{Q}$, and that f has a simple (removable) discontinuity at each point $x \in \mathbb{Q}$.

Solution. We claim that $\lim_{x \to 0} f(x) = 0$ for all $x \in \mathbb{R}$. To prove the claim, let $x \in \mathbb{R}$, and let $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{N} < \varepsilon$. For $1 \leq n \leq N$, let

$$S_n = \left\{ \frac{a}{n} \colon a \in \mathbb{Z} \text{ and } 0 < \left| \frac{a}{n} - x \right| < 1 \right\}.$$

Then S_n is finite; in fact, $\operatorname{card}(S_n) \leq 2n$. Set

$$S = \bigcup_{n=1}^{N} S_n,$$

which is a finite union of finite sets and hence finite. Also, $x \notin S$. Set

$$\delta = \min\left(1, \ \min_{y \in S} |y - x|\right).$$

Then $\delta > 0$ because $x \notin S$ and S is finite.

Let $y \in \mathbb{R}$ satisfy $0 < |y - x| < \delta$. If $y \notin \mathbb{Q}$, then $|f(y) - 0| = 0 < \varepsilon$. Otherwise, because $y \notin S$, $|y - x| < 1$, and $y \neq x$, it is not possible to write $y = \frac{p}{q}$ with $q \leq N$. Thus, when we write $y = \frac{p}{q}$ in lowest terms, we have $q > N$, so $f(y) = \frac{1}{q}$ satisfies $0 < f(y) < \frac{1}{N} < \varepsilon$. This shows that $|f(y) - 0| < \varepsilon$ in this case also.

The claim immediately implies that f is continuous at all points x for which $f(x) = 0$ and has a removable discontinuity at every x for which $f(x) \neq 0$. $\square$