Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Book, notes, calculators, and all other electronic devices are prohibited. The only allowed materials are blank paper and pens or pencils.

Unless otherwise specified, I will follow the usual conventions on the symbols used in limits at \( \infty \). That is, I will assume that \( \lim_{n \to \infty} f(n) = L \) refers to the limit as \( n \) runs through positive integers, and that \( \lim_{x \to \infty} f(x) = L \) refers to the limit as \( x \) runs through real numbers. If, in your solution to some problem, you don’t specify, I will make some assumption depending on the notation and context, and there will be no appeal except in cases in which I actually misread something.

Write your name and your student ID on your paper.

Total: 120 points, plus extra credit. Time: 120 minutes.

1. (12 points/part; total 36 points.) Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required.

   (a) Let \( a, b \in \mathbb{R} \) satisfy \( a < b \). If a sequence \( (f_n)_{n \in \mathbb{Z}_{>0}} \) of continuous functions \( f_n: (a, b) \to \mathbb{R} \) converges pointwise to a function \( f: (a, b) \to \mathbb{R} \), but \( f \) is not continuous on \( (a, b) \), then the convergence is not uniform.

   (For the construction of possible counterexamples, you may use the standard properties of exponential and trigonometric functions and their inverses from elementary calculus.)

   (b) Let \( f: (0, \infty) \to \mathbb{R} \) be a continuous function. Then the improper integral \( \int_{0}^{\infty} f(t) \, dt = \lim_{x \to \infty} \int_{0}^{x} f(t) \, dt \) (where \( x \) runs through positive real numbers) exists if and only if \( \lim_{n \to \infty} \int_{0}^{n} f(t) \, dt \) (where \( n \) runs through positive integers) exists.

   (For the construction of possible counterexamples, you may use the standard properties of exponential and trigonometric functions and their inverses from elementary calculus.)

   (c) Let \( [a, b] \subset \mathbb{R} \) be a closed bounded interval. Let \( f: [a, b] \to \mathbb{R} \) be Riemann integrable. Then there is a partition \( P \) of \([a, b]\) such that \( U(P, f) = L(P, f) \).

   For the construction of possible counterexamples, you may use the standard properties of exponential and trigonometric functions and their inverses from elementary calculus.

2. (10 points) Compute \( \lim_{x \to 0^+} \frac{1}{x} \int_{0}^{x} \exp(1 + t^2) \, dt \).

   You may use the standard properties of exponential and trigonometric functions from elementary calculus.
3. (30 points) Let \( a, b \in \mathbb{R} \) satisfy \( a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a nondecreasing function. For \( x \in [a, b] \) define \( F(x) = \int_a^x f \). Let \( x_0 \in (a, b) \), and suppose that \( F'(x_0) \) exists. Prove that \( F'(x_0) = f(x_0) \).

4. (25 points) For \( x \in \mathbb{R} \) define
\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\sqrt{n}x).
\]
Prove that \( f \) is continuously differentiable. Justify all steps. Be sure to make clear which theorems you are using. You may use the standard properties of exponential and trigonometric functions from elementary calculus.

5. (19 points) Let \( f : \mathbb{R} \to \mathbb{C} \) be uniformly continuous. For \( n \in \mathbb{Z}_{>0} \) define \( f_n : \mathbb{R} \to \mathbb{C} \) by \( f_n(x) = f(x + \frac{1}{n}) \) for \( x \in \mathbb{R} \). Prove that the functions \( f_n \) converge uniformly to \( f \) on \( \mathbb{R} \).

Extra Credit. (Don’t do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

(25 extra credit points.) Let \( a, b \in \mathbb{R} \) satisfy \( a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a nondecreasing function. For \( x \in [a, b] \) define \( F(x) = \int_a^x f \). Let \( x_0 \in (a, b) \), and suppose that \( F'(x_0) \) exists. Prove that \( f \) is continuous at \( x_0 \).

(The hypotheses are the same as in Problem 3, and you may use the result of that problem here.)