Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework.

Book, notes, calculators, and all other electronic devices are prohibited. The only allowed materials are blank paper and pens or pencils.

Unless otherwise specified, I will follow the usual conventions on the symbols used in limits at $\infty$. That is, I will assume that $\lim_{n \to \infty} f(n) = L$ refers to the limit as $n$ runs through positive integers, and that $\lim_{x \to \infty} f(x) = L$ refers to the limit as $x$ runs through real numbers. If, in your solution to some problem, you don’t specify, I will make some assumption depending on the notation and context, and there will be no appeal except in cases in which I actually misread something.

Write your name and your student ID on your paper.

Total: 120 points, plus extra credit. Time: 120 minutes.

1. (12 points/part; total 36 points.) Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required.

(a) Let $a, b \in \mathbb{R}$ satisfy $a < b$. If a sequence $(f_n)_{n \in \mathbb{Z}^+}$ of continuous functions $f_n: (a, b) \to \mathbb{R}$ converges pointwise to a function $f: (a, b) \to \mathbb{R}$, but $f$ is not continuous on $(a, b)$, then the convergence is not uniform.

Solution. True. According to Theorem 7.12 of Rudin’s book, the uniform limit of continuous functions is continuous. \( \Box \)

(b) Let $f: (0, \infty) \to \mathbb{R}$ be a continuous function. Then the improper integral $\int_0^\infty f(t) \, dt = \lim_{x \to \infty} \int_0^x f(t) \, dt$ (where $x$ runs through positive real numbers) exists if and only if $\lim_{n \to \infty} \int_0^n f(t) \, dt$ (where $n$ runs through positive integers) exists.

Solution. False. Let $f(x) = \pi \cos(\pi x)$ for $x \in \mathbb{R}$. Then $\int_0^x f(t) \, dt = \sin(\pi x)$ for all $x \in \mathbb{R}$. Thus $\int_0^n f(t) \, dt = 0$ for all $n \in \mathbb{Z}^+$, so, with the limit running over $n \in \mathbb{Z}^+$, we have $\lim_{n \to \infty} \int_0^n f(t) \, dt = 0$. However, with the limit running over $n \in \mathbb{Z}^+$,

$$\lim_{n \to \infty} \int_0^{n+1/2} f(t) \, dt = \lim_{n \to \infty} f \left( n + \frac{1}{2} \right) = 1,$$

so, with the limit running over $x \in (0, \infty)$, $\lim_{x \to \infty} f(x)$ does not exist. \( \Box \)
One does not need trigonometric functions.

**Alternate solution (sketch).** Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
    f(x) = \begin{cases} 
    x - n & n \in \mathbb{Z} \text{ and } n - \frac{1}{4} \leq x < n + \frac{1}{4} \\
    n + \frac{1}{2} - x & n \in \mathbb{Z} \text{ and } n + \frac{1}{4} \leq x < n + \frac{3}{4}.
    \end{cases}
\]

This function is continuous, and is also a counterexample. \( \square \)

Since one of the claimed implications is false, the statement is false. There is therefore no need to say that the other claimed implications is true.

(c) Let \( [a, b] \subset \mathbb{R} \) be a closed bounded interval. Let \( f : [a, b] \to \mathbb{R} \) be Riemann integrable. Then there is a partition \( P \) of \( [a, b] \) such that \( U(P, f) = L(P, f) \).

For the construction of possible counterexamples, you may use the standard properties of exponential and trigonometric functions and their inverses from elementary calculus.

**Solution.** The statement is false. Take \( a = 0, b = 1 \), and

\[
    f(x) = \begin{cases} 
    0 & 0 \leq x < 1 \\
    1 & x = 1
    \end{cases}
\]

for all \( x \in [a, b] \). Then \( f \) is Riemann integrable because \( f \) is monotone. (This is Theorem 6.9 of Rudin’s book.) Let \( P = (x_0, x_1, \ldots, x_n) \) be a partition of \([0,1]\), so that \( 0 = x_0 < x_1 < \cdots < x_n = 1 \). Then \( L(P, f) = 0 \) and \( U(P, f) = x_n - x_{n-1} > 0 \). This completes the solution. \( \square \)

**Alternate solution.** The statement is false. Take \( a = 0, b = 1 \), and \( f(x) = x \) for all \( x \in [a, b] \). Then \( f \) is Riemann integrable because \( f \) is continuous. (This is Theorem 6.8 of Rudin’s book.) Let \( P = (x_0, x_1, \ldots, x_n) \) be a partition of \([0,1]\), so that \( 0 = x_0 < x_1 < \cdots < x_n = 1 \). Then

\[
    L(P, f) = \sum_{j=1}^{n} x_{j-1}(x_j - x_{j-1}) \quad \text{and} \quad U(P, f) = \sum_{j=1}^{n} x_j(x_j - x_{j-1}).
\]

These are not equal, because

\[
    U(P, f) - L(P, f) = \sum_{j=1}^{n} (x_j - x_{j-1})^2 \geq (x_1 - x_0)^2,
\]

which is nonzero because \( x_1 - x_0 > 0 \). \( \square \)

2. (10 points) Compute \( \lim_{x \to 0^+} \frac{1}{x} \int_0^x \exp(1 + t^2) \, dt \).

You may use the standard properties of exponential and trigonometric functions from elementary calculus.

**Solution.** For \( x \in [0, \infty) \), set

\[
    F(x) = \int_0^x \exp(1 + t^2) \, dt.
\]

Then, since \( F(0) = 0 \), we are to compute

\[
    \lim_{x \to 0^+} \frac{1}{x} F(x) = \lim_{h \to 0^+} \frac{F(h) - F(0)}{h} = F'(0).
\]
By the Fundamental Theorem of Calculus, and using continuity of \( t \mapsto \exp(1 + t^2) \), we get \( F'(0) = \exp(1 + 0^2) = e. \) □

Alternate solution (outline). Use L’Hospital’s Rule. □

L’Hospital’s Rule is overkill, since the limit in the problem is just the definition of the right-hand derivative at 0 of the function

\[
F(x) = \int_0^x \exp(1 + t^2) \, dt.
\]

Second alternate solution. For \( t \in [0, 1] \) we have \( 0 \leq t^2 \leq t \), so

\[
e \leq \exp(1 + t^2) \leq \exp(1 + t).
\]

Therefore, for \( x \in [0, 1] \),

\[
ex = \int_0^x e \, dt \leq \int_0^x \exp(1 + t^2) \, dt \leq \int_0^x \exp(1 + t) \, dt = \exp(1 + x) - e,
\]

and for \( x \in (0, 1] \),

\[
\frac{1}{x} \int_0^x \exp(1 + t^2) \, dt \leq \frac{1}{x} \int_0^x \exp(1 + t) \, dt = \frac{\exp(1 + x) - e}{x}.
\]

With \( g(x) = \exp(1 + x) \), we have

\[
\lim_{x \to 0^+} \frac{\exp(1 + x) - e}{x} = g'(0) = e.
\]

(This can also be done using L’Hospital’s Rule, but that is overkill.) Since \( \lim_{x \to 0^+} = e \) as well, it follows from (1) that

\[
\lim_{x \to 0^+} \frac{1}{x} \int_0^x \exp(1 + t^2) \, dt = e.
\]

This completes the solution. □

This solution is not recommended for this problem, because it is much easier to use the Fundamental Theorem of Calculus. However, the method is important elsewhere.

3. (30 points) Let \( a, b \in \mathbb{R} \) satisfy \( a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a nondecreasing function. For \( x \in [a, b] \) define \( F(x) = \int_a^x f \). Let \( x_0 \in (a, b) \), and suppose that \( F'(x_0) \) exists. Prove that \( F'(x_0) = f(x_0) \).

Solution. We first claim that \( F'(x_0) \geq f(x_0) \). To prove the claim, let \( x \in (x_0, b] \). Then, using \( f(t) \geq f(x_0) \) when \( t \in [x_0, x] \) at the second step,

\[
\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(t) \, dt \geq \frac{1}{x - x_0} \int_{x_0}^x f(x_0) \, dt = f(x_0).
\]

Therefore

\[
F'(x_0) = \lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} \geq f(x_0).
\]

The claim is proved.

We next claim that \( F'(x_0) \leq f(x_0) \). To prove the claim, let \( x \in [a, x_0] \). Then, using \( f(t) \leq f(x_0) \) when \( t \in [x, x_0] \) at the second step,

\[
\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x}^{x_0} f(t) \, dt \leq \frac{1}{x - x_0} \int_{x}^{x_0} f(x_0) \, dt = f(x_0).
\]
Therefore
\[ F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} \leq f(x_0). \]
The claim is proved.
The two claims clearly imply \( F'(x_0) = f(x_0). \)

4. (25 points) For \( x \in \mathbb{R} \) define
\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\sqrt{n}x). \]
Prove that \( f \) is continuously differentiable. Justify all steps. Be sure to make clear which theorems you are using. You may use the standard properties of exponential and trigonometric functions from elementary calculus.

Solution. For \( n \in \mathbb{Z}_{>0} \) and \( x \in \mathbb{R} \), set \( f_n(x) = \frac{1}{n} \sin(\sqrt{n}x). \) Then \( f'_n(x) = \frac{1}{n} \cos(\sqrt{n}x). \) Since \( |\cos(t)| \leq 1 \) for all \( t \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \), the Weierstrass \( M \)-Test implies that \( \sum_{n=1}^{\infty} f'_n(x) \) converges uniformly on \( \mathbb{R} \). Since \( f'_n \) is continuous for all \( n \in \mathbb{Z}_{>0} \), the sum is moreover a continuous function. Also, \( \sum_{n=1}^{\infty} f_n(0) \) converges (to 0). Using Theorem 7.17 of Rudin on the sequences of partial sums, we find that, on any compact interval \([a, b] \) containing 0, the series \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly to a function \( f_{a,b} \) such that \( f'_{a,b}(x) = \sum_{n=1}^{\infty} f'_n(x) \) for all \( x \in [a, b] \). Obviously \( f_{a,b} = f|_{[a,b]} \). Therefore \( f'(x) = \sum_{n=1}^{\infty} f'_n(x) \) for all \( x \in \mathbb{R} \). Since the series on the right converges to a continuous function, this completes the proof.

5. (19 points) Let \( f : \mathbb{R} \to \mathbb{C} \) be uniformly continuous. For \( n \in \mathbb{Z}_{>0} \) define \( f_n : \mathbb{R} \to \mathbb{C} \) by \( f_n(x) = f \left( x + \frac{1}{n} \right) \) for \( x \in \mathbb{R} \). Prove that the functions \( f_n \) converge uniformly to \( f \) on \( \mathbb{R} \).

Solution. Let \( \epsilon > 0 \). Since \( f \) is uniformly continuous on \( \mathbb{R} \), there is \( \delta > 0 \) such that whenever \( x, y \in \mathbb{R} \) satisfy \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \). Choose \( N \in \mathbb{Z}_{>0} \) so large that \( \frac{1}{N} < \delta \). For \( x \in \mathbb{R} \) and \( n \geq N \), we then have \( |(x + \frac{1}{n}) - x| < \delta \) (because \( \frac{1}{n} < \delta \)), so
\[ |f_n(x) - f(x)| = \left| (x + \frac{1}{n}) - x \right| < \epsilon. \]
This completes the solution.

Extra Credit. (Don’t do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

(25 extra credit points.) Let \( a, b \in \mathbb{R} \) satisfy \( a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a nondecreasing function. For \( x \in [a, b] \) define \( F(x) = \int_a^x f \). Let \( x_0 \in (a, b) \), and suppose that \( F'(x_0) \) exists. Prove that \( f \) is continuous at \( x_0 \).

(The hypotheses are the same as in Problem 3, and you may use the result of that problem here.)

Solution. Suppose that \( f \) is not continuous at \( x_0 \). Since \( f \) is nondecreasing, we must have
\[ \lim_{x \to x_0^+} f(x) > f(x_0) \quad \text{or} \quad \lim_{x \to x_0^-} f(x) < f(x_0). \]
Suppose $\lim_{x \to x_0^+} f(x) > f(x_0)$. Set $z = \lim_{x \to x_0^+} f(x)$. Then for all $x \in (x_0, b]$ and all $t \in (x_0, x]$, we have $f(t) \geq z$. Since $\int_{x_0}^{x} [z - f(x_0)] \chi_{\{x_0\}} = 0$, it follows that
\[ \int_{x_0}^{x} f(t) \, dt \geq \int_{x_0}^{x} z \, dt = (x - x_0)z. \]
Therefore
\[ \frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^{x} f(t) \, dt \geq z > f(x_0). \]
So $F'(x_0) \neq f(x_0)$. This contradicts the result of Problem 3.

Now suppose $\lim_{x \to x_0^-} f(x) < f(x_0)$. Set $z = \lim_{x \to x_0^-} f(x)$. Then for all $x \in [a, x_0]$ and all $t \in [x, x_0]$, we have $f(t) \leq z$. Since $\int_{x_0}^{x} [z - f(x_0)] \chi_{\{x_0\}} = 0$, it follows that
\[ \int_{x}^{x_0} f(t) \, dt \leq \int_{x}^{x_0} z \, dt = (x_0 - x)z. \]
Therefore
\[ \frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x_0 - x} \int_{x}^{x_0} f(t) \, dt \leq z < f(x_0). \]
It follows that $F'(x_0) \neq f(x_0)$. Again, this contradicts the result of Problem 3. □