MATHEMATICS 414 [514]: FALL QUARTER, 2018

Course Number: Math 414 and Math 514.
Course Title: Introduction to Analysis 1.
Class Time and Place: MWF 11–11:50 am in 303 Deady. Discussion section: Tu 11–11:50 am, room to be announced.

I expect be away on a research visit during the 7th week of classes, maybe into the 8th week. The lectures will probably be given by substitutes.

Prerequisites: Math 413 or Math 513.
CRN: 24367 (Math 414); 24376 (Math 514).
Instructor: N. Christopher Phillips.
Office: 105 Deady.
Office hours: MF 2:30–3:20 pm, Th 11–11:50 am, or by appointment. (Office hours are subject to change.)

Please knock. I have found it necessary to keep my office door closed, usually even during office hours, because of my location. Otherwise I get asked all the time where to find nonexistent rooms or the Math Department office.

Email: See my home page or the course website. The subject line of your message should start with “M414” (without the quotation marks), followed by your last name, then first initial.

I will use email occasionally to distribute general announcements. I will give short replies to emailed questions. I don’t type at a reasonable speed, so I will rarely answer complicated questions by email. Please come to office hours instead.

When emailing me, please use plain text (7 bit ASCII) only. That is, only the characters found on a standard English language keyboard; no curved quotation marks, curved apostrophes, accented letters, Greek letters, etc. Use TEX pseudocode for mathematical symbols, including Greek letters. Don’t send html; for security and privacy reasons, I don’t read html email. Don’t use MIME or other encoding. Send binary “attachments” only by prior arrangement. Never send Microsoft Word files—I can’t read them.

Course Website: http://pages.uoregon.edu/ncp/Courses/Math414_W19_Web/Math414_W19_Web.html, or go to my home page (https://pages.uoregon.edu/ncp/) and follow the link to Math 414[514].

Copies of some course materials will be posted here.

Course Description: The three quarter sequence Math 413–Math 415 gives a rigorous presentation of calculus of one and several variables (the material of Math 251–Math 253 and Math 281–Math 282). In Math 414, we will cover differentiation, Riemann integration, and continuity, differentiation, and integration of sequences and series of functions. We will also give rigorous constructions of the standard functions from elementary calculus.

Date: 7 January 2019.
courses, and possibly some others (such as the Gamma function). We will look briefly at Fourier series.


**Notes on Terminology:** The book contains a few oddities of notation and terminology which I will not follow. These are the same as for last quarter, except that I have added a paragraph about the term “countable”.

Open intervals are referred to as “segments”. The sets of real numbers and rational numbers are called $R$ and $Q$ instead of the more usual $\mathbb{R}$ and $\mathbb{Q}$. (In particular, I will feel free to use the letters $R$ and $Q$ for other purposes.) In contrast to this, the book suddenly switches from $f$ to $F$ when the function is supposed to take values in $\mathbb{R}^n$ instead of a general metric space; I will make no such distinction.

To avoid the perpetual confusion about whether $0 \in \mathbb{N}$, in anything formal (including homework and solutions) I will take $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$ and $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. These symbols are unfortunately ugly, but they are unambiguous.

To avoid confusion about whether, say, an increasing sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{R}$ is supposed to satisfy $x_{n+1} > x_n$ or $x_{n+1} \geq x_n$, in anything formal I will refer only to “nondecreasing” and “strictly increasing” sequences (as well as functions, etc.). Similarly, I will use “nonincreasing” and “strictly decreasing”. To see the problem, compare Rudin’s definition of a monotonically increasing sequence with the definition of an increasing sequence in the textbook we are currently using in Math 251–253. For similar reasons, I will avoid “positive” in favor of “nonnegative” and “strictly positive”.

I will take “countable” to include finite subsets. Rudin (following a number of authors) defines a set $E$ to be “countable” if there is a bijection from $E$ to $\mathbb{Z}_{>0}$. Other authors also consider finite sets to be countable. (At least one book uses “countable” for one version and “denumerable” for the other.) Rudin’s choice is poor: many of the authors (including Rudin) who exclude finite sets in their definition of “countable” nevertheless implicitly include finite sets in practice. One sees this, for example, in Rudin’s Problem 2.22. Everybody agrees that a metric space with only one point is separable. But, formally, Rudin’s definition excludes such spaces: they don’t have countable dense subsets because they don’t have infinite subsets. (A careful author would have written “at most countable”.) The wording “a metric space is separable if it has a countable dense subset” (and in similar related definitions) is nearly universal.

**Grading:**

**Homework:** Contribution to final grade: approximately 20%. Due every Monday, except as otherwise stated.

The homework will be hard, some of it quite hard, but it is an essential part of the course.

The homework consists of proofs. This means that everything must be proved. In particular, if a problem asks for an example or counterexample, you must prove that your example has the required properties. If a problem asks whether something is true, you must not only decide whether it is, but also provide a proof or counterexample.
as appropriate; just as above, counterexamples must be proved to have
the required properties.

Proofs should be clear, correct, and complete. They should use
complete sentences and correct punctuation. They should have all
quantifiers in the right places. They should be in logical order. For
example, for a proof of convergence of a sequence from the definition,
start with “Let $\varepsilon > 0$” and end with the statement that something
is less than $\varepsilon$. If there are subsidiary arguments in a proof, say what
is being claimed, say that you are proving it, prove it, and say when
the proof of your claim is done. A reader who can’t follow any of the
mathematics should nevertheless be able to easily tell where the proof
of your subsidiary claim starts and where it ends.

I expect to post on the course website at least sketches of some
solutions after the homework is turned in.

Some procedural points:

(1) Staple pages together. Don’t fold or tear corners or fold in half.

(2) If you cooperate with someone else on any particular problem,
you must acknowledge the cooperation (including the name of
the person) in your solution.

**One midterm:** Contribution to final grade: approximately 30%. Date:
Monday 11 February (week 6). The midterm from the last time I
taught the course is on the course website, along with solutions. (The
format of the midterm this time may be a bit different.) A large portion
of the midterm will ask for proofs which you will have to devise for
yourself during the exam period. This midterm was given during a
two hour period outside of class.

I am willing to arrange to give the entire exam during a longer
period outside of the normal class time. Such a procedure can only be
followed if suitable rooms and times can be found, and if everybody
enrolled in the class agrees to it.

**Final Examination:** Contribution to final grade: approximately 50%.
Date: Monday 18 March, 10:15 am–12:15 pm, in our usual classroom.

No early final exams, according to University rules. If you have
another final exam scheduled at the same time as our final exam, you
need to give me the details (course number and instructor) by Monday
of the 7th week of classes.

Similar comments on the content apply as for the midterm.

I keep originals of the final exam, but will give out on request a
scan or copy of your exam.

**Accessible Education Center:** Students with documented learning dis-
abilities who wish to use the Accessible Education Center (linked on
the course website) to take tests under specifically arranged conditions
should let me know as soon as possible, certainly by Wednesday of the
third week of classes. Such students must also be sure to meet the Ac-
cessible Education Center’s separate deadlines for requests; these are
likely to be a week or more in advance of the exam date (much more
for final exams). I can’t do anything to help a student who misses its
deadline. (I have tried in the past.)
Grading Complaints: Complaints about the grading of any exam must be submitted in writing by the beginning of the first class period after the class in which that exam is returned.

Extra Credit: Extra credit will be awarded to the first person to detect any error or misprint in the book, in homework solutions, in exams and exam solutions, or on the course website. More extra credit will be given for catching mathematical errors than misspelled words or wrong dates. To get the extra credit, you must tell me what the correct version should be.

There are likely to be a number of mistakes to catch. When going through materials from the last time I taught this course, I found a number of misprints, including a few egregious ones, which were not caught despite a similar offer for extra credit. (A particularly bad one in a homework solution for the fall quarter was “not compact” for “compact”.)

The following misprints and mistakes in the book have already been caught, so can’t be reused.

- Nonmathematical misprint: Page 4, second paragraph under 1.9: “largest” misspelled.
- Multiple mathematical errors: Rudin defines a set to be countable if it has the same cardinality as $\mathbb{Z}_{>0}$. In problems 2.22–2.26, this gives the wrong definition of the terms “separable” and “countable base”. In particular, Problem 2.25 would be false as stated: a one point space is compact, but could not be separable because it wouldn’t have any countable subsets.
- Mathematical error: Problem 5.13 should have $|x|^a$ in the definition of $f$.

Also, I decide the order of quotation marks and punctuation by meaning, which is not the usual convention. (However, you can get extra credit for finding an example in which my choice doesn’t follow my convention.)

Learning Outcomes: The course deals with:

- Differentiability of functions, differentiable functions, and their derivatives, for functions with domain a suitable subset of $\mathbb{R}$ and codomain $\mathbb{R}^n$ or $\mathbb{C}^n$.
- Riemann and Riemann-Stieltjes integrability of functions, integrable functions, and their integrals, for functions with domain an interval in $\mathbb{R}$ and codomain $\mathbb{R}^n$ or $\mathbb{C}^n$.
- Pointwise and uniform convergence of sequences and series of functions with domain a metric space and codomain $\mathbb{R}^n$ and $\mathbb{C}^n$. Continuity, integrability, and differentiability of the limit function under suitable hypotheses on the type of convergence, domain of the function, and the functions in the sequence or series.
- Special properties of power series.
- Equicontinuity and the Arzela-Ascoli Theorem.
- The Stone-Weierstrass Theorem for compact metric spaces.
• Rigorous construction of the standard functions found in elementary calculus courses, together with rigorous proofs of their properties (continuity, calculation of derivatives, functional equations, etc.).

• Some basic properties of Fourier series.

The successful student will be able to precisely state the definitions associated with these topics, and will be able to rigorously prove standard facts. The successful student will also be able to rigorously prove or disprove (as appropriate) statements about these topics which have not been encountered before, and which are at a level of difficulty appropriate to a mid-level introductory analysis course. The successful student will be able to write these proofs so that they are correct, complete, clear, readable, and in logical order.

Academic Conduct: The code of student conduct and community standards is linked on the course website. In this course, it is appropriate to help each other on homework as long as the work you are submitting is your own and you understand it, and you give the names of any people you cooperated with. It is not appropriate to help each other while taking exams, to look at other students’ exams while taking exams, or to bring any unauthorized material to exams. The only authorized material is pens and pencils.

Schedule: This schedule is subject to minor changes as may be necessary.

We will cover most of Chapters 5–8 of the book in the winter quarter, possibly getting into Chapter 9. The part of Chapter 5 already done in the previous course (going through L’Hospital’s Rule) won’t be repeated. We will follow the book fairly closely.

The book usually used in Math 316–317 is S. Abbott, *Understanding Analysis*. Much of the material of this quarter is also there, but often restricted to special cases. As a good example from last quarter, continuity is mostly only treated there in the context of functions from subsets of \( \mathbb{R} \) to \( \mathbb{R} \), and we considered continuity of functions between metric spaces. This quarter, we will often take the codomain for differentiable and integrable functions to be \( \mathbb{R}^n \) or \( \mathbb{C}^n \), and we will consider the Riemann-Stieltjes integral instead of just the Riemann integral.

The schedule gives approximate times that topics will be discussed. It is necessary to read the book; not everything will be done in class. Homework assignments are subject to change.

**Week 1 (7–11 Jan.):** Finish Chapter 5 (differentiation). Start Chapter 6 (Riemann-Stieltjes integration).

Homework (due Monday 14 Jan.): Chapter 5: 1, 2, 3, 4, 5, 9, 11, 13. Point values: 10 points per problem, except 4 points per part in Problem 13 and 12 points for Problem 2. Correction in Problem 13: the function is supposed to be

\[
    f(x) = \begin{cases} 
    |x|^a \sin(|x|^{-c}) & x \neq 0 \\
    0 & x = 0.
    \end{cases}
\]

(Rudin has \( x^a \) in place of \( |x|^a \), which is not defined for \( x < 0 \) and most values of \( a \).)
Week 2 (14–18 Jan.): Continue Chapter 6 (Riemann-Stieltjes integration).

Homework (due Wednesday 23 Jan.): Chapter 5: 12, 22a–c, 26; Chapter 6: 2, 4; Problem A. Point values: 10 points for each problem part. (Chapter 5 Problem 22a–c and Problem A have 3 parts each.)

Problem A: Let $X$ be a complete metric space, and let $f: X \to X$ be a function. Suppose that there is a constant $k$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$.

1) Prove that $f$ is uniformly continuous.
2) Suppose that $k < 1$. Prove that $f$ has a unique fixed point, that is, there is a unique $x \in X$ such that $f(x) = x$.
3) Show that the conclusion in Part (2) need not hold if $X$ is not complete.

Week 3 (21–24 Jan.): Continue Chapter 6 (Riemann-Stieltjes integration). (Short week, due to holiday Monday.)

Homework (due Monday 28 Jan.): Chapter 6: 7, 8, 10, 11. In Problem 10d, do only the improper integrals from Problem 7. Point values: 10 points for each problem part. (Problem 7 has 2 parts and Problem 10 has 4 parts. Total 80 points instead of 100 points.)

Week 4 (28 Jan.–1 Feb.): Finish Chapter 6 (Riemann-Stieltjes integration); start Chapter 7 (sequences and series of functions).

Homework (due Monday 4 Feb.): Chapter 6: 15; Chapter 7: 1, 2, 3, 4. The notation in Problem 15 is not very good: it should have $\int_a^b x[f(x)]^2 \, dx$. Point values: 20 points per problem.

Week 5 (4–8 Feb.): Continue Chapter 7 (sequences and series of functions).

Homework (due Friday 8 Feb.; short assignment, because of the midterm on Monday): Chapter 7: 5, 6, 7. Point values: 13 points each for Problems 5 and 6; 14 points for Problems 7. (Total 40 points instead of 100 points.)


Homework (due Monday 18 Feb.): Chapter 7: 9, 16, 19, 20, 21. The notation in Problem 20 is not very good: it should have $\int_0^b x[f(x)]^2 \, dx$. Point values: 20 points per problem.

Week 7 (18–22 Feb.): Start Chapter 8 (standard elementary functions; Fourier series).

Homework (due Monday 25 Feb.): Chapter 7: 22, 24; Chapter 8: 2, 3, 4, 5. Point values: 5 points per part in Problems 4 and 5; otherwise, 10 points for each of the two other problems in Chapter 8, and 20 points for each of the problems in Chapter 7.

Week 8 (25 Feb.–1 March): (short week, due to holidays): Continue Chapter 8 (standard elementary functions; Fourier series).

Homework (due Monday 4 March): Chapter 8: 6, 7, 8, 9, 10, 11. (There is a hint for Problem 10, but it is on the page after the statement of the problem.) Point values: 10 points for each problem part, except 15 points each for Problems 8 and 11 and 20 points for Problem 10.
**Week 9 (4–8 March):** Continue Chapter 8 (standard elementary functions; Fourier series).

Homework (due Monday 11 March): Chapter 5: 27; Chapter 8: 20, 23, 24, 25. (Also look at Problems 22 and 26.) Point values: 20 points per problem.

**Week 10 (11–15 March):** Finish Chapter 8 (standard elementary functions; Fourier series).

Homework (due Friday 15 March): Chapter 8: 12, 13, 14. Point values: 10 points for each problem part. (Total 70 points instead of 100 points.)

Sa 16 March, 2:00–4:00 pm: Review session for the final exam. Room to be announced.

**Final exams week (18–22 March):** M 18 March, 10:15 am–12:15 pm: final exam, in regular classroom.

**Important dates:** These are not guaranteed, and they are on different days of the quarter than they have been in the past.

- **Su 6 Jan.:** Last day to process a complete drop (100% refund, no W recorded).
- **Sa 12 Jan.:** Last day to drop this course (100% refund, no W recorded; after this date, W’s are recorded).
- **Sa 12 Jan.:** Last day to process a complete drop (90% refund, no W recorded).
- **Su 13 Jan.:** Last day to add this course.
- **Su 13 Jan.:** Last day to withdraw from this course (100% refund, W recorded).
- **W 16 Jan.:** Last day to change to or from audit.
- **Su 20 Jan.:** Last day to withdraw from this course (75% refund, W recorded).
- **M 21 Jan.:** No classes.
- **Su 27 Jan.:** Last day to withdraw from this course (50% refund, W recorded).
- **Su 3 Feb.:** Last day to withdraw from this course (25% refund, W recorded).
- **Su 24 Feb.:** Last day to withdraw from this course (0% refund, W recorded).
- **Su 24 Feb.:** Last day to change grading option or variable credits for this course. (Tuition penalties apply when reducing credits.)