Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework. (Reminder: \( \log(x) \) always means \( \log_e(x) \).)

Unless otherwise specified, I will follow the usual conventions on the symbols used in limits at \( \infty \). For example, I will assume that \( \lim_{n \to \infty} f(n) = L \) refers to the limit as \( n \) runs through positive integers, and that \( \lim_{x \to \infty} f(x) = L \) refers to the limit as \( x \) runs through real numbers. If, in your solution to some problem, you don’t specify, I will make some assumption depending on the notation and context, and there will be no appeal except in cases in which I actually misread something.

Write your name and your student ID on your paper.

Total: 200 points, plus extra credit; time (extended): 150 minutes.

**Problem 1** (10 points/part; total 40 points): Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required. However, most of the credit is for the justification.

(a) Let \( f: (0, 1] \to \mathbb{R} \) be a continuous function. Then \( \lim_{n \to \infty} f \left( \frac{1}{n} \right) \) (as \( n \) runs through \( \mathbb{Z}^+ \)) exists if and only if \( \lim_{x \to 0^+} f(x) \) (as \( x \) runs through \( (0, 1] \)) exists.

(b) Let \((f_n)_{n \in \mathbb{Z}^+}\) be a sequence of Riemann integrable functions \( f_n: [0, 1] \to \mathbb{R} \). Suppose that for every \( r \in (0, \frac{1}{2}) \), we have \( f_n \to 0 \) uniformly on \([r, 1 - r]\). Also assume \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} f_n(1) = 0 \). Then \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0 \).

(c) Let \( f: (-1, 1) \to \mathbb{R} \) have a power series expansion \( \sum_{n=0}^{\infty} a_n x^n \) valid on the interval \((-1, 1)\), and suppose that \( f \left( \frac{-1}{2} \right) = -1 \). Then \( f \) can’t be zero at every point of the set \( \left\{ \frac{1}{2^k} : k = 1, 2, 3, \ldots \right\} \).

(d) The set of all polynomial functions of the form \( p(x) = a_0 + a_1 x^3 + a_2 x^6 + \cdots + a_n x^{3^n} \), for arbitrary \( n \in \mathbb{Z}_{\geq 0} \) and with coefficients \( a_k \in \mathbb{R} \) for \( k = 0, 1, \ldots, n \), is dense in \( C_b([-1, 1]) \), the space of all real valued continuous functions on \([-1, 1]\) with its usual metric (obtained from \( \| \cdot \|_\infty \)).

**Problem 2** (25 points): Let \( f: [0, 1] \to \mathbb{R} \) be a bounded function such that \( f|_{(0,1]} \) is continuous. Without using any theorems on partitions or integration, prove that for every \( \varepsilon > 0 \) there is a partition \( P \) of \([0,1]\) such that \( U(P,f) - L(P,f) < \varepsilon \).

**Problem 3** (25 points): Let \((f_n)_{n \in \mathbb{Z}^+}\) be a sequence of functions \( f_n: \mathbb{R} \to \mathbb{C} \). Suppose there is a function \( f: \mathbb{R} \to \mathbb{C} \) such that \( f_n \to f \) uniformly, and assume

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that \( f \) is uniformly continuous. For \( n \in \mathbb{Z}_{>0} \) and \( x \in \mathbb{R} \) define \( g_n(x) = f_n(x + \frac{1}{n}) \)
Prove that the sequence \( (g_n)_{n \in \mathbb{Z}_{>0}} \) converges uniformly to \( f \) on \( \mathbb{R} \).

**Problem 4** (25 points): Prove that the sequence \( (g_n) \), with
\[
\sum_{n=0}^{\infty} e^{-nx} \cos(nx)
\]
converges to a continuous function \( g(x) \) on \((0, \infty)\).

**Problem 5** (25 points): Let \( f : [a, b] \to [0, \infty) \) be a continuous function. Prove that
\[
\lim_{t \to \infty} \int_{a}^{b} \frac{f(x)}{f(x) + t} \, dx = 0.
\]
(Here \( t \) runs through all positive real numbers.)

**Problem 6** (10 points/part; total 40 points). In this problem, the only properties of the elementary transcendental functions (exp, log, sin, etc.) you may use are those proved in the text or in the lectures.

(a) Compute \( \lim_{x \to 0^+} \frac{1}{x} \int_{0}^{x} \sqrt{2 + \cos(t^2)} \, dt \).
(b) Let \( a \in \mathbb{R} \). Find, with proof, a power series of the form
\[
\sum_{n=0}^{\infty} c_n (x-a)^n
\]
which converges to \( e^x \) for all \( x \in \mathbb{R} \).
(c) Prove that \( e^x \geq 1 + x \) for all \( x \in \mathbb{R} \), with equality if and only if \( x = 0 \).
(d) Compute \( \frac{d}{dx} \left( \int_{-\pi}^{x} \sqrt{17 + t^4 + [\cos(t)]^4} \, dt \right) \).

**Problem 7** (10 points/part; total 20 points). Let \( f : \mathbb{R} \to \mathbb{C} \) be the \( 2\pi \) periodic function such that \( f(x) = |x| \) for \(-\pi \leq x \leq \pi\).

(a) Find the Fourier coefficients of \( f \).
(b) Use the result of Part (a) to evaluate
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}.
\]

**Extra credit.** (Don’t do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

Prove that for every odd integer \( n \geq 1 \), we have
\[
e^x > \sum_{k=0}^{n} \frac{x^k}{k!}
\]
for all real \( x \) with \( x < 0 \).
(The statement is also true for $x > 0$, but no extra credit points will be earned for proving this.)