MATH 414[514] WINTER 2002 FINAL EXAM SOLUTIONS

Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework. (Reminder: \( \log(x) \) always means \( \log_e(x) \).)

Unless otherwise specified, I will follow the usual conventions on the symbols used in limits at \( \infty \). For example, I will assume that \( \lim_{n \to \infty} f(n) = L \) refers to the limit as \( n \) runs through positive integers, and that \( \lim_{x \to \infty} f(x) = L \) refers to the limit as \( x \) runs through real numbers. If, in your solution to some problem, you don’t specify, I will make some assumption depending on the notation and context, and there will be no appeal except in cases in which I actually misread something.

Write your name and your student ID on your paper.

Total: 200 points, plus extra credit; time (extended): 150 minutes.

Problem 1 (10 points/part; total 40 points): Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required. However, most of the credit is for the justification.

(a) Let \( f : (0, 1] \to \mathbb{R} \) be a continuous function. Then \( \lim_{n \to \infty} f \left( \frac{1}{n} \right) \) (as \( n \) runs through \( \mathbb{Z}^>0 \)) exists if and only if \( \lim_{x \to 0^+} f(x) \) (as \( x \) runs through \( (0, 1] \)) exists.

Solution. False. Take \( f(x) = \sin \left( \frac{\pi}{x} \right) \). Then \( f \left( \frac{1}{n} \right) = 0 \) for all \( n \in \mathbb{Z}^>0 \), so \( \lim_{n \to \infty} f \left( \frac{1}{n} \right) = 0 \), but \( \lim_{x \to 0^+} f(x) \) does not exist.

Details (not required for the solution): Set \( x_n = \left( 2n + \frac{1}{2} \right)^{-1} \) for \( n \in \mathbb{Z}^>0 \). Then \( f(x_n) = 1 \) for all \( n \in \mathbb{Z}^>0 \). Since \( \left( \frac{1}{n} \right)_{n \in \mathbb{Z}^>0} \) and \( (x_n)_{n \in \mathbb{Z}^>0} \) are two sequences in \( (0, 1] \) which converge to 0, and since \( \lim_{n \to \infty} f \left( \frac{1}{n} \right) \neq \lim_{n \to \infty} f(x_n) \), it follows that \( \lim_{x \to 0^+} f(x) \) does not exist. \( \square \)

(b) Let \( (f_n)_{n \in \mathbb{Z}^>0} \) be a sequence of Riemann integrable functions \( f_n : [0, 1] \to \mathbb{R} \). Suppose that for every \( r \in (0, \frac{1}{2}) \), we have \( f_n \to 0 \) uniformly on \( [r, 1-r] \). Also assume \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} f_n(1) = 0 \). Then

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0.
\]

Solution. False. For \( n \in \mathbb{Z}^>0 \) set

\[
f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \text{ or } x = 0. \end{cases}
\]

For every \( r \in (0, \frac{1}{2}) \), for all large enough \( n \), the function \( f_n|_{[r, 1-r]} \) is the constant function with value zero, so \( f_n \to 0 \) uniformly on \( [r, 1-r] \). Clearly \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} f_n(1) = 0. \)

However, \[ \int_0^1 f_n(x) \, dx = 1 \]
for all \( n \), so \[ \lim_{n \to \infty} \int_0^1 f_n(x) \, dx \neq 0. \]

Details (not required for the solution): Let \( r \in (0, \frac{1}{2}) \); we show that \( f_n \to 0 \) uniformly on \([r, 1 - r]\). Let \( \varepsilon > 0 \). Choose \( N \in \mathbb{Z}_{>0} \) so large that \( \frac{1}{N} < r \). For \( n \geq N \) and \( x \in [r, 1 - r] \), we have \( f_n(x) = 0 \), so that \( |f_n(x) - 0| < \varepsilon \).

Since for all \( n \in \mathbb{Z}_{>0} \) we have \( f_n(0) = f_n(1) = 0 \), it is clear that \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} f_n(1) = 0. \)

\( \square \)

Alternate solution. False. For \( n \in \mathbb{Z}_{>0} \) set \( f_n(x) = \begin{cases} n^2 & 0 < x < \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \text{ or } x = 0. \end{cases} \)

For every \( r \in (0, \frac{1}{2}) \), for all large enough \( n \), the function \( f_n_{|[r, 1-r]} \) is the constant function with value zero, so \( f_n \to 0 \) uniformly on \([r, 1-r]\). Clearly \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} f_n(1) = 0. \)

However, \[ \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \lim_{n \to \infty} n = \infty. \]

This completes the solution. \( \square \)

(c) Let \( f: (-1, 1) \to \mathbb{R} \) have a power series expansion \( \sum_{n=0}^{\infty} a_n x^n \) valid on the interval \((-1, 1)\), and suppose that \( f\left(-\frac{1}{2}\right) = -1 \). Then \( f \) can’t be zero at every point of the set \( \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\} \).

Solution. True. Let \( S = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\} \). Suppose \( f(x) = 0 \) for all \( x \in S \). Since the set \( S \) has a limit point in the open interval of convergence \((-1, 1)\), the function \( f \) must be the zero function. This contradicts \( f\left(-\frac{1}{2}\right) = -1. \)

Details (not required for the solution): The conclusion that \( f \) must be the zero function follows from Theorem 8.5 of Rudin’s book. \( \square \)

(d) The set of all polynomial functions of the form \( p(x) = a_0 + a_1 x^3 + a_2 x^6 + \cdots + a_n x^{3n} \),
for arbitrary \( n \in \mathbb{Z}_{\geq0} \) and with coefficients \( a_k \in \mathbb{R} \) for \( k = 0, 1, \ldots, n \), is dense in \( C_{\mathbb{R}}([-1, 1]) \), the space of all real valued continuous functions on \([-1, 1]\) with its usual metric (obtained from \( \| \cdot \|_{\infty} \)).

Solution. Let \( A \subset C_{\mathbb{R}}([-1, 1]) \) be the set of all polynomial functions of the form described. Then \( A \) is a subalgebra of \( C_{\mathbb{R}}([-1, 1]) \) which contains the constant functions. Moreover, \( A \) separates the points of \([-1, 1]\), because the function \( p(x) = x^3 \) by itself separates the points of \([-1, 1]\). The Stone-Weierstrass Theorem therefore implies that \( A \) is dense in \( C_{\mathbb{R}}([-1, 1]) \).

Details (not required for the solution): It is obvious that \( A \) is a real vector space which contains the constant functions.
We claim that $A$ is closed under multiplication. Let $p, q \in A$. Then there are $m, n \in \mathbb{Z}_{\geq 0}$ and $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in \mathbb{R}$ such that
\[
p(x) = a_0 + a_1 x^3 + a_2 x^6 + \cdots + a_m x^{3m}
\]
and
\[
q(x) = b_0 + b_1 x^3 + b_2 x^6 + \cdots + b_n x^{6n}
\]
for all $x \in [-1, 1]$. For $l = 0, 1, \ldots, m + n$, define
\[
c_l = \sum_{k=\max(0,l-n)}^{\min(m,l)} a_k b_{l-k}.
\]
Then
\[
p(x)q(x) = c_0 + c_1 x^3 + c_2 x^6 + \cdots + c_{m+n} x^{3(m+n)}
\]
for all $x \in [-1, 1]$. This shows that $pq \in A$. The claim is proved. \[\square\]

Alternate solution. Let $A \subset C_\mathbb{R}([-1,1])$ be the set of all polynomial functions of the form described. We show that $A$ contains all polynomials. The Weierstrass Theorem will then imply that $A$ is dense in $C_\mathbb{R}([-1,1])$, from which it will follow that $A$ is dense in $C_\mathbb{R}([-1,1])$.

Let
\[
p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n
\]
be a polynomial function on $[-1, 1]$. Let $\varepsilon > 0$. Define
\[
g(x) = a_0 + a_1 \sqrt[3]{x} + a_2 (\sqrt[3]{x})^2 + \cdots + a_n (\sqrt[3]{x})^n.
\]
Then $g \in C_\mathbb{R}([-1, 1])$. The Weierstrass Theorem provides a polynomial function
\[
q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m
\]
such that $\|q - g\|_\infty < \varepsilon$. Since $x \mapsto x^3$ is a bijection from $[-1, 1]$ to itself, it follows that
\[
\sup_{x \in [-1,1]} |g(x^3) - g(x^3)| < \varepsilon.
\]
The function $x \mapsto q(x^3)$ is in $A$, since it is
\[
q(x) = b_0 + b_1 x^3 + b_2 x^6 + \cdots + b_m x^{3m}.
\]
Also, $g(x^3) = p(x)$. Since $\varepsilon > 0$ is arbitrary, we have shown that $p \in A$, as desired. \[\square\]

Problem 2 (25 points): Let $f: [0,1] \to \mathbb{R}$ be a bounded function such that $f|_{[0,1]}$ is continuous. Without using any theorems on partitions or integration, prove that for every $\varepsilon > 0$ there is a partition $P$ of $[0,1]$ such that $U(P,f) - L(P,f) < \varepsilon$.

Solution. Let $\varepsilon > 0$. Set $M = \sup_{t \in [0,1]} |f(t)|$. Set $t_1 = \min\left(\frac{1}{2}, \frac{1}{4\varepsilon}(M+1)^{-1}\right)$. Then $f$ is continuous on $[t_1,1]$. Since this interval is compact, $f$ is uniformly continuous on $[t_1,1]$. Therefore there is $\delta > 0$ such that whenever $r, s \in [t_1,1]$ satisfy $|r-s| < \delta$, then $|f(r) - f(s)| < \frac{\varepsilon}{2}$. Choose $n \in \mathbb{Z}_{>0}$ such that $(n-1)\delta > 1$. For $k = 2, 3, \ldots, n$ define
\[
t_k = t_1 + \frac{(k-1)(1-t_1)}{n-1}.
\]
Also set $t_0 = 0$. Then $0 = t_0 < t_1 < \cdots < t_n = 1$. Let $P$ be the partition $P = (t_0, t_1, \ldots, t_n)$ of $[0,1]$. We claim that $U(P,f) - L(P,f) < \varepsilon$. 

We prove the claim. By rearranging terms, we get
\[ U(P, f) - L(P, f) = \sum_{j=1}^{n} \left( \sup_{t \in [t_{j-1}, t_j]} f(t) - \inf_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1}). \]

To estimate the term with \( j = 1 \), we observe that
\[ \sup_{t \in [t_0, t_1]} f(t) \leq M \quad \text{and} \quad \inf_{t \in [t_0, t_1]} f(t) \geq -M. \]

Also, \( t_1 - t_0 \leq \frac{1}{4} \varepsilon (M + 1)^{-1} \). Therefore
\[ \left( \sup_{t \in [t_0, t_1]} f(t) - \inf_{t \in [t_0, t_1]} f(t) \right) (t_1 - t_0) \leq 2M \cdot \frac{1}{4} \varepsilon (M + 1)^{-1} < \frac{\varepsilon}{2}. \]

To estimate the other terms, we observe that if \( j \geq 1 \) then
\[ t_j - t_{j-1} = \frac{1 - t_1}{n - 1} < \frac{1}{n - 1} < \delta. \]
Therefore
\[ \left( \sup_{t \in [t_{j-1}, t_j]} f(t) - \inf_{t \in [t_{j-1}, t_j]} f(t) \right) \leq \sup_{s, t \in [t_{j-1}, t_j]} |f(s) - f(t)| \leq \frac{\varepsilon}{2}. \]

It follows that
\[ \sum_{j=2}^{n} \left( \sup_{t \in [t_{j-1}, t_j]} f(t) - \inf_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1}) \leq \frac{\varepsilon}{2} \sum_{j=2}^{n} (t_j - t_{j-1}) = \frac{\varepsilon}{2} (1 - t_1) < \frac{\varepsilon}{2}. \]

Combining our two estimates, we get
\[ U(P, f) - L(P, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

The claim is proved. \( \Box \)

**Problem 3** (25 points): Let \((f_n)_{n \in \mathbb{Z}_{>0}}\) be a sequence of functions \(f_n : \mathbb{R} \to \mathbb{C}\). Suppose there is a function \(f : \mathbb{R} \to \mathbb{C}\) such that \(f_n \to f\) uniformly, and assume that \(f\) is uniformly continuous. For \(n \in \mathbb{Z}_{>0}\) and \(x \in \mathbb{R}\) define \(g_n(x) = f_n \left( x + \frac{1}{n} \right) \)

Prove that the sequence \((g_n)_{n \in \mathbb{Z}_{>0}}\) converges uniformly to \(f\) on \(\mathbb{R}\).

**Solution.** Let \(\varepsilon > 0\). Since \(f\) is uniformly continuous on \(\mathbb{R}\), there is \(\delta > 0\) such that whenever \(x, y \in \mathbb{R}\) satisfy \(|x - y| < \delta\), then \(|f(x) - f(y)| < \frac{\varepsilon}{2}\). Choose \(N_1 \in \mathbb{Z}_{>0}\) so large that \(\frac{1}{N_1} < \delta\). Since \(f_n \to f\) uniformly, we can choose \(N_2 \in \mathbb{Z}_{>0}\) so large that whenever \(n \geq N_2\) and \(x \in \mathbb{R}\) then \(|f_n(x) - f(x)| < \frac{\varepsilon}{2}\). Set \(N = \max(N_1, N_2)\). For \(x \in \mathbb{R}\) and \(n \geq N\), we then have \(|(x + \frac{1}{n}) - x| = \frac{1}{n} < \delta\), so
\[ |g_n(x) - f(x)| \leq |f_n \left( x + \frac{1}{n} \right) - f \left( x + \frac{1}{n} \right)| + |f \left( x + \frac{1}{n} \right) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

This completes the solution. \( \Box \)

Warning: There is no valid proof which proceeds directly via the estimate
\[ |g_n(x) - f(x)| \leq |f_n \left( x + \frac{1}{n} \right) - f_n(x)| + |f_n(x) - f(x)|. \]
One would need to show that for $\varepsilon > 0$ there is $\delta > 0$ such that whenever $n \in \mathbb{Z}_{>0}$ and $x, y \in \mathbb{R}$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. This is uniform equicontinuity, and proving it requires some variant of the argument given in the solution above. Solutions which appear to merely assume this to be true are incorrect, and will receive very little credit.

**Problem 4** (25 points): Prove that the series
\[
\sum_{n=0}^{\infty} e^{-nx} \cos(nx)
\]
converges to a continuous function $g(x)$ on $(0, \infty)$.

**Solution.** We first claim that for $x \in (0, \infty)$, the series converges. To prove the claim, we use the Comparison Test. Since $x > 0$, we have $e^{-x} < 1$, so that
\[
\sum_{n=0}^{\infty} |e^{-nx} \cos(nx)| \leq \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} (e^{-x})^n = \frac{1}{1 - e^{-x}} < \infty.
\]

Therefore the series converges absolutely for all $x > 0$. The claim follows, and we can define
\[
g(x) = \sum_{n=0}^{\infty} e^{-nx} \cos(nx)
\]
for $x > 0$.

Now we prove continuity. Let $x_0 \in (0, \infty)$. To prove that $g$ is continuous at $x_0$, it suffices to show that the restriction of $g$ to some open interval $I$ containing $x_0$ is continuous. We take $I = \left(\frac{1}{2}x_0, \infty\right)$. For $x \in I$, we have $x > \frac{1}{2}x_0$, so that $e^{-nx} < e^{-nx_0}/2$, and, at the second step using (1) with $\frac{1}{2}x_0$ in place of $x$, we get
\[
\sum_{n=0}^{\infty} \sup_{x \in I} |e^{-nx} \cos(nx)| \leq \sum_{n=0}^{\infty} e^{-nx_0}/2 = \frac{1}{1 - e^{-nx_0}/2} < \infty.
\]

The Weierstrass test therefore implies that the series for $g(x)$ converges uniformly on $I$. Since all the terms are continuous functions, this implies that $g$ is continuous on $I$. □

**Warning:** The series for $g(x)$ does not converge uniformly on $(0, \infty)$. Proofs which claim that it does are wrong, and will receive little credit.

**Alternate solution.** We calculate as follows, for $x > 0$. The last step is the formula for the sum of a geometric series, and it is justified since
\[
|e^{(i-1)x}| = |e^{(-i-1)x}| = e^{-x},
\]
and $e^{-x} < 1$ for $x > 0$. We have
\[
\sum_{n=0}^{\infty} e^{-nx} \cos(nx) = \sum_{n=0}^{\infty} e^{-nx} \cdot \frac{1}{2} (e^{inx} + e^{-inx}) = \frac{1}{2} \sum_{n=0}^{\infty} e^{n(i-1)x} + \frac{1}{2} \sum_{n=0}^{\infty} e^{n(-i-1)x} = \frac{1}{2} \sum_{n=0}^{\infty} (e^{(i-1)x})^n + \frac{1}{2} \sum_{n=0}^{\infty} (e^{(-i-1)x})^n = \frac{1}{2(1 - e^{(i-1)x})} + \frac{1}{2(1 - e^{(-i-1)x})}.
\]
The final expression is a continuous function of $x$ for $x \in (0, \infty)$. □

**Problem 5** (25 points): Let $f: [a, b] \to [0, \infty)$ be a continuous function. Prove that

$$\lim_{t \to \infty} \int_a^b \frac{f(x)}{f(x) + t} \, dx = 0.$$  

(Here $t$ runs through all positive real numbers.)

**Solution 1.** Let $\varepsilon > 0$. Since $[a, b]$ is compact and $f$ is continuous, there is $M < \infty$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. We may assume $M > 0$. Set $T = \frac{2M(b-a)}{\varepsilon} < \infty$.

For any $t > T$ and $x \in [a, b]$, we have

$$|f(x)| \leq M \quad \text{and} \quad f(x) + t \geq T > 0,$$

whence

$$\frac{f(x)}{f(x) + t} \leq \frac{M}{T} = \frac{\varepsilon}{2(b-a)}.$$

Therefore $t > T$ implies

$$0 \leq \int_a^b \frac{f(x)}{f(x) + t} \, dx \leq \frac{\varepsilon}{2(b-a)} (b-a) < \varepsilon.$$

This completes the solution. □

*Note:* A solution which only proves that

$$\lim_{n \to \infty} \int_a^b \frac{f(x)}{f(x) + n} \, dx = 0,$$

where $n$ runs through positive integers, is not a proof and will lose many points. However, one can successfully use this idea if one is careful, as shown in the next two solutions.

**Solution 2 (outline).** Prove that for every sequence $(t_n)$ in $(0, \infty)$ with $t_n \to \infty$, the functions

$$g_n(x) = \frac{f(x)}{f(x) + t_n}$$

for $n \in \mathbb{Z}_{>0}$, converge uniformly to zero. Then use the result on convergence of the integrals of a uniformly convergent sequence of functions.

**Solution 3 (outline).** Prove that the functions

$$g_n(x) = \frac{f(x)}{f(x) + n}$$

for $n \in \mathbb{Z}_{>0}$, converge uniformly to zero. Use the result on convergence of the integrals of a uniformly convergent sequence of functions to show that

$$\lim_{n \to \infty} \int_a^b \frac{f(x)}{f(x) + n} \, dx = 0,$$

where $n$ runs through positive integers. Next, observe that if $0 < s \leq t$ then

$$\frac{f(x)}{f(x) + s} \geq \frac{f(x)}{f(x) + t}.$$
for all $x \in [a, b]$, whence
\[ \int_a^b \frac{f(x)}{f(x) + s} \, dx \geq \int_a^b \frac{f(x)}{f(x) + t} \, dx. \]

We now know that
\[ t \mapsto \int_a^b \frac{f(x)}{f(x) + t} \, dx \]
is a nonincreasing function, and that its values at integers converge to 0 at infinity. From this, it is easy to prove that its values at real numbers converge to 0 at infinity. \qed

Finally, we give a solution which uses more machinery but demonstrates what is really happening here.

\textit{Solution 4 (outline).} Define $F: (0, \infty) \to C_R([a, b])$ by

\[ F(t)(x) = \frac{f(x)}{f(x) + t}. \]

Prove that $F$ is continuous and that $\lim_{t \to \infty} F(t) = 0$ (in $C_R([a, b])$, that is, that $\lim_{t \to \infty} \|F(t)\|_\infty = 0$). Then prove that integration, as a function from $C_R([a, b])$ to $\mathbb{R}$, is continuous. \qed

\textbf{Problem 6} (10 points/part; total 40 points). In this problem, the only properties of the elementary transcendental functions (exp, log, sin, etc.) you may use are those proved in the text or in the lectures.

(a) Compute $\lim_{x \to 0^+} \frac{1}{x} \int_0^x \sqrt{2 + \cos(t^2)} \, dt$.

\textit{Solution.} Set
\[ F(x) = \int_0^x \sqrt{2 + \cos(t^2)} \, dt. \]

Then, since $F(0) = 0$, we are to compute
\[ \lim_{x \to 0^+} \frac{1}{x} F(x) = \lim_{h \to 0^+} \frac{F(h) - F(0)}{h} = F'(0). \]

By the Fundamental Theorem of Calculus, and using continuity of the function $t \mapsto \sqrt{2 + \cos(t^2)}$, we get $F'(0) = \sqrt{2 + \cos(0^2)} = \sqrt{3}$. \qed

(b) Let $a \in \mathbb{R}$. Find, with proof, a power series of the form
\[ \sum_{n=0}^{\infty} c_n (x - a)^n \]
which converges to $e^x$ for all $x \in \mathbb{R}$.

\textit{Solution 1.} We know the series expansion
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]
holds for all \( x \in \mathbb{R} \) (even for all \( x \in \mathbb{C} \)). Theorem 8.3 of Rudin therefore implies that the series
\[
\sum_{n=0}^{\infty} c_n(x-a)^n,
\]
with \( c_n = \frac{1}{n!} \exp^{(n)}(a) \) for \( n \in \mathbb{Z}_{\geq 0} \), converges to \( e^x \) for all \( x \in \mathbb{R} \). Using the (known) derivative of \( e^x \), we get \( \exp^{(n)}(a) = e^a \) for all \( n \geq 0 \). This gives
\[
e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n
\]
for all \( x \in \mathbb{R} \). \( \square \)

**Solution 2.** For \( x \in \mathbb{C} \), we have
\[
e^x = \exp(x-a+a) = \exp(x-a) \exp(a).
\]
The definition
\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}
\]
for \( z \in \mathbb{C} \) therefore gives the series
\[
e^x = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n
\]
for all \( x \in \mathbb{C} \), and in particular for all \( x \in \mathbb{R} \). \( \square \)

(c) Prove that \( e^x \geq 1 + x \) for all \( x \in \mathbb{R} \), with equality if and only if \( x = 0 \).

**Solution.** Set \( g(x) = e^x - 1 - x \) for all \( x \in \mathbb{R} \). Then \( g'(x) = e^x - 1 \) for all \( x \in \mathbb{R} \). Since \( x \mapsto e^x \) is strictly increasing, we get \( g'(x) < 0 \) for \( x < 0 \) and \( g'(x) > 0 \) for \( x > 0 \). The Mean Value Theorem (actually, one of its corollaries) now implies that \( g \) is strictly decreasing on \((-\infty, 0)\) and strictly increasing on \((0, \infty)\). Since \( g \) is continuous and \( g(0) = 0 \), it follows that \( g(x) > 0 \) for \( x \neq 0 \). The desired inequality is now immediate. \( \square \)

A solution that ignores the case \( x < 0 \) gets at most half credit.

**Alternate solution.** We consider three cases.

First suppose \( x \geq 0 \). Then
\[
e^x - (1+x) = \sum_{n=2}^{\infty} \frac{x^n}{n!}.
\]
The right hand side is nonnegative, because every term is nonnegative, and is zero if and only if \( x = 0 \).

Next suppose \( x \leq -1 \). Then \( e^x > 0 \geq 1 + x \). This proves the desired inequality, and equality never holds.

Finally, suppose \(-1 < x < 0\). Since the power series converges absolutely, we may arrange it as
\[
e^x - (1+x) = \sum_{n=2}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \left( 1 + \frac{x}{2n+1} \right).
\]
Each term in the last sum is strictly positive, because for \( n \in \mathbb{Z}_{>0} \) and \( x \in (-1,0) \) we have
\[
\frac{x^{2n}}{(2n)!} > 0 \quad \text{and} \quad \left| \frac{x^{2n}}{2n+1} \right| = \frac{|x|}{2n+1} < 1.
\]
Therefore the sum is strictly positive, whence \( e^x > 1 + x \). \( \square \)

(d) Compute \( \frac{d}{dx} \left( \int_{-\pi}^{x} \sqrt[4]{17 + t^4 + |\cos(t)|^4} \, dt \right) \).

Solution. For \( x \in \mathbb{R} \), set \( f(x) = \sqrt[4]{17 + x^4 + |\cos(x)|^4}, \quad F(x) = \int_{-\pi}^{x} f(t) \, dt, \quad \) and \( g(x) = x^4. \)

Then we are to compute \((F \circ g)'(x)\). Using the Chain Rule, the Fundamental Theorem of Calculus, and continuity of \( t \mapsto \sqrt[4]{17 + t^4 + |\cos(t)|^4} \),
\[
(F \circ g)'(x) = f'(g(x))g'(x) = f(g(x))g'(x) = \sqrt[4]{17 + x^4 + |\cos(x)|^4} \cdot 4x^3 = 4x^3 \sqrt[4]{17 + x^4 + |\cos(x)|^4}.
\]
This completes the solution. \( \square \)

Problem 7 (10 points/part; total 20 points). Let \( f : \mathbb{R} \to \mathbb{C} \) be the \( 2\pi \) periodic function such that \( f(x) = |x| \) for \( -\pi \leq x \leq \pi \).

(a) Find the Fourier coefficients of \( f \).

Solution. The Fourier coefficients of \( f \) are given by
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-inx} \, dx
\]
for \( n \in \mathbb{Z} \). In particular,
\[
c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \left( \frac{1}{\pi} \right) \left( \frac{x^2}{2} \right) = \frac{\pi}{2}.
\]
If \( n \neq 0 \), then we calculate (some justifications given afterwards):
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-inx} \, dx = \frac{1}{2\pi} \left( \int_{-\pi}^{0} (-x) e^{-inx} \, dx + \int_{0}^{\pi} x e^{-inx} \, dx \right)
\]
\[
= \frac{1}{2\pi} \left( \int_{0}^{\pi} x e^{inx} \, dx + \int_{0}^{\pi} x e^{-inx} \, dx \right) = \frac{1}{2\pi} \int_{0}^{\pi} 2x \cos(nx) \, dx
\]
\[
= \frac{1}{2\pi} \left[ \frac{2}{n} x \sin(nx) + \frac{2}{n^2} \cos(nx) \right]_{0}^{\pi} = \frac{1}{\pi n^2} (\cos(n\pi) - 1).
\]
The third step comes from a change of variable in the first integral. The fifth step is from integration by parts; the justification is that
\[
\frac{d}{dx} \left( \frac{2}{n} x \sin(nx) + \frac{2}{n^2} \cos(nx) \right) = 2x \cos(nx).
\]
Now \( \cos(n\pi) = 1 \) for \( n \) even and \( \cos(n\pi) = -1 \) for \( n \) odd. So we get, including the separate computation for \( n = 0 \):

\[
c_n = \begin{cases} 
\frac{\pi}{2} & n = 0 \\
-\frac{2}{\pi n} & n \text{ odd} \\
0 & n \text{ even and nonzero}.
\end{cases}
\]

This completes the solution.

(b) Use the result of Part (a) to evaluate

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.
\]

**Solution.** Parseval’s Theorem tells us that

\[
\sum_{n=-\infty}^{\infty} c_n^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.
\]

The right hand side is

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \left( \frac{1}{2\pi} \right) \left( \frac{2\pi^3}{3} \right) = \frac{\pi^2}{3}.
\]

Using the fact that \( c_{-n} = c_n \) for all \( n \in \mathbb{Z} \), and that \( c_n = 0 \) for \( n \) even and nonzero, we find that the left hand side is

\[
\left( \frac{\pi}{2} \right)^2 + 2 \sum_{n=0}^{\infty} \left( -\frac{2}{\pi(2n+1)} \right)^2 = \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.
\]

Equating and solving for the sum, we get

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.
\]

This completes the solution.

**Extra credit.** (Don’t do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

Prove that for every odd integer \( n \geq 1 \), we have

\[
e^x > \sum_{k=0}^{n} \frac{x^k}{k!}
\]

for all real \( x \) with \( x < 0 \).

(The statement is also true for \( x > 0 \), but no extra credit points will be earned for proving this.)

**Solution.** We prove by induction on \( n \) that for \( n = 0, 1, 2, \ldots \) we have

\[
e^x > \sum_{k=0}^{2n+1} \frac{x^k}{k!}
\]

for all real \( x \) with \( x < 0 \). The case \( n = 0 \) is in Problem 6(c).
Now suppose the result is known for $n$. For $x \in \mathbb{R}$, set

$$g(x) = e^x - \sum_{k=0}^{2n+2} \frac{x^k}{k!}.$$ 

Then

$$g'(x) = e^x - \sum_{k=0}^{2n+1} \frac{x^k}{k!}.$$ 

The induction hypothesis gives $g'(x) > 0$ for all $x < 0$. Therefore the Mean Value Theorem (actually, one of its corollaries) implies that $g$ is strictly increasing on $(-\infty, 0)$. Since $g$ is continuous and $g(0) = 0$, it follows that $g(x) < 0$ for $x < 0$.

Next, for $x \in \mathbb{R}$, set

$$h(x) = e^x - \sum_{k=0}^{2n+3} \frac{x^k}{k!}.$$ 

Then

$$h'(x) = e^x - \sum_{k=0}^{2n+2} \frac{x^k}{k!} = g(x).$$ 

So $h'(x) < 0$ for all $x < 0$. Therefore the Mean Value Theorem (actually, one of its corollaries) implies that $h$ is strictly decreasing on $(-\infty, 0)$. Since $h$ is continuous and $h(0) = 0$, it follows that $h(x) > 0$ for $x < 0$. This is the desired result for $n+1$, so the induction is complete. □