Instructions: All claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework. (Reminder: \( \log(x) \) always means \( \log_e(x) \).)

Unless otherwise specified, I will follow the usual conventions on the symbols used in limits at \( \infty \). For example, I will assume that \( \lim_{n \to \infty} f(n) = L \) refers to the limit as \( n \) runs through positive integers, and that \( \lim_{x \to \infty} f(x) = L \) refers to the limit as \( x \) runs through real numbers. If, in your solution to some problem, you don’t specify, I will make some assumption depending on the notation and context, and there will be no appeal except in cases in which I actually misread something.

Write your name and your student ID on your paper.

Total: 120 points, plus extra credit; time (extended): 110 minutes.

**Problem 1** (12 points/part; total 36 points): Decide whether the following assertions are true or false. Give a brief justification or counterexample; complete proofs, and complete proofs of counterexamples, are not required. However, most of the credit is for the justification.

(a) Let \( a, b \in \mathbb{R} \) with \( a < b \). If \( f: [a, b] \to \mathbb{R} \) is a function such that \(|f|\) is Riemann integrable, then \( f \) is Riemann integrable.

(b) A sequence of discontinuous functions can not converge uniformly to a continuous function.

(c) Let \((f_n)_{n \in \mathbb{Z}^+}\) be a sequence of functions from \([a, b]\) to \( \mathbb{R} \). If \((f_n)_{n \in \mathbb{Z}^+}\) converges uniformly on \((a, b)\), and if the sequences \((f_n(a))_{n \in \mathbb{Z}^+}\) and \((f_n(b))_{n \in \mathbb{Z}^+}\) converge, then \((f_n)_{n \in \mathbb{Z}^+}\) converges uniformly on \([a, b]\).

**Problem 2** (24 points): Let \((f_n)_{n \in \mathbb{Z}^+}\) be a sequence in \( C([a, b]) \). Suppose that there is a function \( f: [a, b] \to \mathbb{R} \) such that \( f_n \to f \) pointwise on \([a, b]\) and uniformly on every closed interval contained in \((a, b)\). Prove carefully that \( f \) is continuous on \((a, b)\).

**Problem 3** (30 points): For \( x \in \mathbb{R} \) define

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nx).
\]

Calculate \( f'(0) \) or show that \( f'(0) \) does not exist. Justify all steps. (Be sure to make clear which theorems you are using. You may use the standard properties of trigonometric functions from elementary calculus.)

**Problem 4** (10 points/part; total 30 points):

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Date: 14 Feb. 2002.
(a) Compute \( \frac{d}{dx} \left( \int_{-17}^{x} \sin \left( \sqrt[3]{q^2 + 17} \right) \, dq \right) \) for \( x > -17 \). (You may use the standard properties of trigonometric functions from elementary calculus.)

(b) Define \( f : [0, 1] \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0 & x < 1 \\
1 & x = 1. 
\end{cases}
\]
Prove directly from the definition that \( f \) is Riemann integrable.

(c) Compute \( \frac{d}{dx} \left( \int_{0}^{x^2} \sqrt{1 + t^2 + e^t} \, dt \right) \) for \( x > 0 \). (You may use the standard properties of the exponential function from elementary calculus.)

**Extra credit.** (Don’t do this problem until you have checked your work on all the others. This problem will be counted only if you get at least 70% on the rest of the exam.)

Let \( X \) be a compact metric space, and let \( E \) the set of functions \( f : X \to \mathbb{C} \) such that
\[
L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}
\]
is finite. For \( f, g \in E \), define
\[
\rho(f, g) = L(f - g) + \|f - g\|_{\infty}.
\]

(a) (5 extra credit points.) Prove that \( \rho \) is a metric on \( E \).

(b) (40 extra credit points.) Prove that \( E \) is a complete metric space.

(c) (15 extra credit points.) Prove that \( E \) is a dense subset of \( C(X) \) in the usual metric on \( C(X) \).