Problem 5.1: Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Prove that $f$ is constant.

Solution. We first prove that $f'(x) = 0$ for all $x \in \mathbb{R}$. For $h \in \mathbb{R} \setminus \{0\}$, we have

$$\left| \frac{f(x + h) - f(x)}{h} \right| = \frac{|f(x + h) - f(x)|}{|h|} \leq \frac{|h|^2}{|h|} = |h|.$$

It follows immediately that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = 0.$$

Since this is true for all $x \in \mathbb{R}$, it now follows from Theorem 5.11(b) of Rudin’s book that $f$ is constant.

The solution above is the intended solution. However, there is another solution which is nearly as easy and does not use calculus.

Alternate solution. Let $x, y \in \mathbb{R}$ and let $\varepsilon > 0$; we prove that $|f(x) - f(y)| < \varepsilon$. It will clearly follow that $f$ is constant.

Choose $N \in \mathbb{Z}_{>0}$ with $N > \varepsilon^{-1}(x - y)^2$. The hypothesis implies that, for any $k$, we have

$$|f \left( x + (k - 1) \cdot \frac{1}{N} (y - x) \right) - f \left( x + k \cdot \frac{1}{N} (y - x) \right) | \leq \left( \frac{1}{N} \right)^2 \left( \frac{(y - x)^2}{N} \right) < \frac{1}{N} \cdot \varepsilon.$$

Therefore

$$|f(x) - f(y)| \leq \sum_{k=1}^{N} |f \left( x + (k - 1) \cdot \frac{1}{N} (y - x) \right) - f \left( x + k \cdot \frac{1}{N} (y - x) \right) | < N \cdot \frac{1}{N} \cdot \varepsilon = \varepsilon.$$

This completes the solution. 

Problem 5.2: Let \( f: (a, b) \to \mathbb{R} \) satisfy \( f'(x) > 0 \) for all \( x \in (a, b) \). Prove that \( f \) is strictly increasing, that its inverse function \( g \) is differentiable, and that

\[
g'(f(x)) = \frac{1}{f'(x)}
\]

for all \( x \in (a, b) \).

Solution. That \( f \) is strictly increasing on \((a, b)\) follows from the Mean Value Theorem and the fact that \( f'(x) > 0 \) for all \( x \in (a, b) \).

Define

\[ c = \inf_{x \in (a, b)} f(x) \quad \text{and} \quad d = \sup_{x \in (a, b)} f(x). \]

(Note that \( c \) could be \(-\infty\) and \( d \) could be \(\infty\).) Our next step is to prove that \( f \) is a bijection from \((a, b)\) to \((c, d)\). Clearly \( f \) is injective, and has range contained in \([c, d]\). If \( c = f(x) \) for some \( x \in (a, b) \), then there is \( q \in (a, b) \) with \( q < x \). This would imply \( f(q) < c \), contradicting the definition of \( c \). So \( c \) is not in the range of \( f \). Similarly \( d \) is not in the range of \( f \). So the range of \( f \) is contained in \((c, d)\). For surjectivity, let \( y_0 \in (c, d) \). By the definitions of \( \inf \) and \( \sup \), there are \( r, s \in (a, b) \) such that \( f(r) < y_0 < f(s) \). Clearly \( r < s \). The Intermediate Value Theorem provides \( x_0 \in (r, s) \) such that \( f(x_0) = y_0 \). This shows that the range of \( f \) is all of \((c, d)\), and completes the proof that \( f \) is a bijection from \((a, b)\) to \((c, d)\).

Now we show that \( g: (c, d) \to (a, b) \) is continuous. Again, let \( y_0 \in (c, d) \), and choose \( r \) and \( s \) as in the previous paragraph. Since \( f \) is strictly increasing, and again using the Intermediate Value Theorem, we see that \( f|_{[r,s]} \) is a continuous bijection from \([r, s]\) to \([f(r), f(s)]\). Since \([r, s]\) is compact, the function \((f|_{[r,s]})^{-1} = g|_{f(r), f(s)}\) is continuous. Since \( y_0 \in (f(r), f(s)) \), it follows that \( g \) is continuous at \( y_0 \). Thus \( g \) is continuous.

Now we find \( g' \). Fix \( x_0 \in (a, b) \), and set \( y_0 = f(x_0) \). For \( y \in (c, d) \setminus \{y_0\} \), we write

\[
\frac{g(y) - g(y_0)}{y - y_0} = \left( \frac{f(g(y)) - f(x_0)}{g(y) - x_0} \right)^{-1}.
\]

(Note that \( g(y) \neq x_0 \) because \( g \) is injective.) Since \( g \) is continuous, we have \( \lim_{y \to y_0} g(y) = x_0 \). Therefore

\[
\lim_{y \to y_0} \frac{f(g(y)) - f(x_0)}{g(y) - x_0} = f'(x_0).
\]

Hence

\[
\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.
\]

That is, \( g'(y_0) \) exists and is equal to \( \frac{1}{f'(x_0)} \), as desired.

I believe, but have not checked, that further use of the Intermediate Value Theorem can be substituted for the use of compactness in the proof that \( g \) is continuous.

Problem 5.3: Let \( g: \mathbb{R} \to \mathbb{R} \) be a differentiable function such that \( g' \) is bounded. Prove that there is \( r > 0 \) such that for all \( \varepsilon \in (0, r) \), the function \( f(x) = x + \varepsilon g(x) \) is injective.
Solution. Set $M = \max\{0, \sup_{x \in \mathbb{R}} [-g'(x)]\}$. Set $r = \frac{1}{M}$. (Take $r = \infty$ if $M = 0$.) Suppose $0 < \varepsilon < r$, and define $f(x) = x + \varepsilon g(x)$ for $x \in \mathbb{R}$. For $x \in \mathbb{R}$, we have

$$f'(x) = 1 + \varepsilon g'(x) = 1 - \varepsilon(-g'(x)) \geq 1 - \varepsilon M > 1 - rM = 0$$

(except that $1 - rM = 1$ if $M = 0$). Thus $f'(x) > 0$ for all $x$, so the Mean Value Theorem implies that $f$ is strictly increasing. In particular, $f$ is injective. \hfill \Box

Problem 5.4: Let $C_0, C_1, \ldots, C_n \in \mathbb{R}$. Suppose

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0.$$  

Prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real solution in $(0, 1)$.

Solution. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_{n-1} x^n}{n} + \frac{C_n x^{n+1}}{n+1}$$

for $x \in \mathbb{R}$. Then $f(0) = 0$ (this is trivial) and $f(1) = 0$ (this follows from the hypothesis). Since $f$ is differentiable on all of $\mathbb{R}$, the Mean Value Theorem provides $x \in (0, 1)$ such that $f'(x) = 0$. Since $f'(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n$,

this is the desired conclusion. \hfill \Box

Problem 5.5: Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and suppose that $\lim_{x \to \infty} f'(x) = 0$. Prove that $\lim_{x \to \infty} |f(x + 1) - f(x)| = 0$.

Solution. Let $\varepsilon > 0$. Choose $M \in \mathbb{R}$ such that $x > M$ implies $|f'(x)| < \varepsilon$. Let $x > M$. By the Mean Value Theorem, there is $z \in (x, x + 1)$ such that $f(x + 1) - f(x) = f'(z)$. Then $|f(x + 1) - f(x)| = |f'(z)| < \varepsilon$. This shows that $\lim_{x \to \infty} |f(x + 1) - f(x)| = 0$. \hfill \Box

Problem 5.9: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f'(x)$ exists for all $x \neq 0$, and that $\lim_{x \to 0} f'(x) = 3$. Does it follow that $f'(0)$ exists?

Solution. We prove that $f'(0) = 3$. Define $g(x) = x$ for $x \in \mathbb{R}$. Then

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 3$$

by assumption. Since $f - f(0)$ vanishes at 0 and has derivative $f'$, Theorem 5.13 of Rudin (L’Hospital’s rule) applies to the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{g(x)}$$

Thus

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 3.$$  

In particular, $f'(0)$ exists. \hfill \Box
It is mathematically bad practice (although it is, unfortunately, tolerated in freshman calculus courses) to write
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 3
\]
before checking that
\[
\lim_{x \to 0} \frac{f'(x)}{g'(x)}
\]
exists, because the equality
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}
\]
is only known to hold when the second limit exists.

**Problem 5.11:** Let \( f \) be a real valued function defined on a neighborhood of \( x \in \mathbb{R} \). Suppose that \( f''(x) \) exists. Prove that
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x).
\]
Show by example that the limit might exist even if \( f''(x) \) does not exist.

**Solution (sketch).** Check using algebra that
\[
\lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h} = f''(x).
\]
Now use Theorem 5.13 of Rudin (L’Hospital’s rule) to show that
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h} = f''(x).
\]
For the counterexample, take
\[
f(t) = \begin{cases} 
1 & t > x \\
0 & t = x \\
-1 & t < x.
\end{cases}
\]
Then
\[
\frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = 0
\]
for all \( h \in \mathbb{R} \setminus \{0\} \). This shows that the limit can exist even if \( f \) isn’t even continuous at \( x \).

Note 1: I gave a counterexample for an arbitrary value of \( x \), but it suffices to give one at a single value of \( x \), such as \( x = 0 \).

Note 2: A legitimate counterexample must be defined at \( x \), since it must satisfy all the hypotheses except for the existence of \( f''(x) \).

Note 3: Another choice for the counterexample is
\[
f(t) = \begin{cases} 
(t - x)^2 & t \geq x \\
-(t - x)^2 & t < x.
\end{cases}
\]
This function is continuous at $x$, and even has a continuous derivative on $\mathbb{R}$, but $f''(x)$ doesn’t exist. One can also construct examples which are continuous nowhere on $\mathbb{R}$.

Note 4: It is tempting to use L’Hospital’s rule a second time, to get

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = \lim_{h \to 0} \frac{f''(x + h) + f''(x - h)}{2}.$$ 

This reasoning is not valid, since the second limit need not exist. (We do not assume that $f''$ is continuous.)

Problem 5.13: Let $a$ and $c$ be fixed real numbers, with $c > 0$, and define $f = f_{a,c} : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} |x|^a \sin(|x|^{-c}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove the following statements. (You may use the standard facts from elementary calculus about the functions $\sin(x)$ and $\cos(x)$.)

1. $f$ is continuous if and only if $a > 0$.
2. $f'(0)$ exists if and only if $a > 1$.
3. $f'$ is bounded if and only if $a \geq 1 + c$.
4. $f'$ is continuous on $[-1, 1]$ if and only if $a > 1 + c$.
5. $f''(0)$ exists if and only if $a > 2 + c$.
6. $f''$ is bounded if and only if $a \geq 2 + 2c$.
7. $f''$ is continuous on $[-1, 1]$ if and only if $a > 2 + 2c$.

The book has

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

However, unless $a$ is a rational number with odd denominator, this function will not be defined for $x < 0$.

Solution to (1) (sketch). Since $x \mapsto \sin(x)$ is continuous, we need only consider continuity at 0. If $a > 0$, then $\lim_{x \to 0} f(x) = 0$ since $|f(x)| \leq |x|^a$ and $\lim_{x \to 0} |x|^a = 0$.

Now define sequences $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$ by

1. $x_n = \frac{1}{\left( (2n + \frac{1}{2}) \pi \right)^{1/c}}$ and $y_n = \frac{1}{\left( (2n + \frac{3}{2}) \pi \right)^{1/c}}$

for $n \in \mathbb{Z}_{>0}$. Then

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0$$

and

$$\sin(|x_n|^{-c}) = 1 \quad \text{and} \quad \sin(|y_n|^{-c}) = -1$$

for all $n \in \mathbb{Z}_{>0}$. (We will use these sequences in other parts of the problem.)

If now $a = 0$, then, with $(x_n)_{n \in \mathbb{Z}_{>0}}$ and $(y_n)_{n \in \mathbb{Z}_{>0}}$ as in (1),

$$\lim_{n \to \infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} f(y_n) = -1,$$
so \( \lim_{x \to 0} f(x) \) does not exist, and \( f \) is not continuous at 0. If \( a < 0 \), then
\[
\lim_{n \to \infty} f(x_n) = \infty \quad \text{and} \quad \lim_{n \to \infty} f(y_n) = -\infty,
\]
with the same result.

Note: Since \( f(0) \) is defined to be 0, we actually need only consider \( \lim_{n \to \infty} f(x_n) \).
The conclusion \( \lim_{x \to 0} f(x) \) does not exist is stronger, and will be useful later.

**Solution to (2) (sketch).** We test for existence of
\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} f_{a-1,c}(h),
\]
which we saw in Part (1) exists if and only if \( a - 1 > 0 \). Moreover (for use below), note that if the limit does exist then it is equal to 0.

**Solution to (3) (sketch).** Boundedness does not depend on \( f'(0) \) (or even on whether \( f'(0) \) exists). So we use the formula
\[
f'(x) = ax^{a-1} \sin (x^{-c}) + cx^{a-c-1} \cos (x^{-c})
\]
for \( x > 0 \), and for \( x < 0 \) we use
\[
f'(x) = -f'(-x) = -f'(|x|) = -a|x|^{a-1} \sin (|x|^{-c}) - c|x|^{a-c-1} \cos (|x|^{-c}).
\]
If \( a - c - 1 \geq 0 \), then also \( a - 1 \geq 0 \) (recall that \( c > 0 \)), and \( f' \) is bounded (by \( c + a \)).

Otherwise, we consider the sequences \((w_n)_{n \in \mathbb{Z}_{>0}}\) and \((z_n)_{n \in \mathbb{Z}_{>0}}\) given by
\[
(2) \quad w_n = \frac{1}{|2n\pi|^{1/c}} \quad \text{and} \quad z_n = \frac{1}{|(2n + 1)\pi|^{1/c}}
\]
for \( n \in \mathbb{Z}_{>0} \). Since
\[
\sin (w_n^{-c}) = \sin (z_n^{-c}) = 0
\]
and
\[
\cos (w_n^{-c}) = 1 \quad \text{and} \quad \cos (z_n^{-c}) = -1,
\]
arguments as in Part (1) show that
\[
\lim_{n \to \infty} f'(w_n) = -\infty \quad \text{and} \quad \lim_{n \to \infty} f'(z_n) = \infty,
\]
so \( f' \) is not bounded.

**Solution to (4) (sketch).** If \( a < 1 + c \), then \( f' \) is not bounded on \([-1, 1] \setminus \{0\}\) by Part (3). Since \([-1, 1] \) is compact, \( f'|[-1,1] \setminus \{0\} \) therefore can’t be the restriction of a continuous function on \([-1, 1] \). If \( a = 1 + c \), then the sequences \((2)\) satisfy
\[
\lim_{n \to \infty} f'(w_n) = -c \quad \text{and} \quad \lim_{n \to \infty} f'(z_n) = c,
\]
so again \( f' \) can’t be the restriction of a continuous function on \([-1, 1] \). If \( a > 1 + c \), then also \( a > 1 \), and \( \lim_{x \to 0} f'(x) = 0 \) by reasoning similar to that of Part (1). Moreover \( f'(0) = 0 \) by the extra conclusion in the proof of Part (2). So \( f' \) is continuous at 0, hence continuous.

**Solution to (5) (sketch).** This is reduced to Part (4) in the same way Part (2) was reduced to Part (1). As there, note also that \( f''(0) = 0 \) if it exists.
Solution to (6) (sketch). For $x \neq 0$, we have
\[ f''(x) = a(a - 1)|x|^{a-2} \sin \left( |x|^{-c} \right) + (2ac - c^2 - c)x^{a-c-2} \cos \left( |x|^{-c} \right) 
- c^2|x|^{a-2c-2} \sin \left( |x|^{-c} \right). \]
(One handles the cases $x > 0$ and $x < 0$ separately, as in Part (3), but this time the resulting formula is the same for both cases.) Since $c > 0$, if $a - 2c - 2 \geq 0$ then also $a > 2 + c$, so $f''$ is bounded on $[-1, 1]\{0\}$. If $a - 2c - 2 < 0$, then, with $(x_n)_{n \in \mathbb{Z} > 0}$ as in (1), we have
\[ f''(x_n) = a(a - 1)x_n^{a-2} - c^2x_n^{a-2c-2} = (a(a - 1)x_n^{a-2} - c^2)x_n^{a-2c-2}. \]
Since $x_n \to 0$, $2c > 0$, and $a - 2c - 2 < 0$, we have
\[ \lim_{n \to \infty} [a(a - 1)x_n^{a-2} - c^2] = -c^2 < 0 \quad \text{and} \quad \lim_{n \to \infty} x_n^{a-2c-2} = \infty. \]
Therefore $\lim_{n \to \infty} f''(x_n) = -\infty$. So $f''$ is not bounded.

Solution to (7) (sketch). Recall from the extra conclusion in Part (5) that $f''(0) = 0$ if it exists. If $a - 2c - 2 > 0$, then also $a - c - 1 > 0$ and $a - 2 > 0$, so $\lim_{x \to 0} f''(x) = 0$ by a more complicated version of the arguments used in Parts (1) and (4). If $a - 2c - 2 < 0$, then $f''$ isn’t bounded on $[-1, 1]\{0\}$, so $f''$ can’t be continuous on $[-1, 1]$. If $a - 2c - 2 = 0$, then $a - 2 > 0$. Therefore
\[ f''(x_n) = a(a - 1)x_n^{a-2} - c^2x_n^{a-2c-2} = a(a - 1)x_n^{a-2} - c^2 \to c^2 \neq 0 \]
as $n \to \infty$. So $f''$ is not continuous at 0.