Generally, a “solution” is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A “solution (sketch)” is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in, perhaps as little as an omitted easy estimate, perhaps almost all the detail.

**Problem 5.12:** Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = |x|^3 \). Compute \( f'(x) \) and \( f''(x) \) for all real \( x \). Prove that \( f'''(0) \) does not exist.

**Solution.** To make clear exactly what is being done, we first prove a lemma.

**Lemma 1.** Let \((a,b) \subset \mathbb{R}\) be an open interval, and let \( f, g : \mathbb{R} \to \mathbb{R} \) be functions. Let \( c \in (a,b) \), and suppose \( f'(c) \) exists. Suppose that there is \( \epsilon > 0 \) such that \((c-\epsilon, c+\epsilon) \subset (a,b)\) and \( g(x) = f(x) \) for all \( x \in (c-\epsilon, c+\epsilon) \). Then \( g'(c) \) exists and \( g'(c) = f'(c) \).

**Proof.** We have
\[
\frac{g(c+h) - g(c)}{h} = \frac{f(c+h) - f(c)}{h}
\]
for all \( h \) with \( 0 < |h| < \epsilon \). Therefore
\[
\lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.
\]
This completes the proof. \( \square \)

Now we start the calculation. For \( x > 0 \), we have \( f(x) = x^3 \). Therefore, by the lemma, \( f'(x) = 3x^2 \) and \( f''(x) = 6x \). Similarly, for \( x < 0 \), we have \( f(x) = -x^3 \), so \( f'(x) = -3x^2 \) and \( f''(x) = -6x \).

The lemma is not useful for \( x = 0 \). So we calculate directly. For \( h \neq 0 \), we have
\[
\left| \frac{f(h) - f(0)}{h} \right| = \frac{|h|^3}{h} = |h|^2.
\]
Therefore
\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0.
\]

With this result in hand, we can calculate \( f''(0) \). For \( h \neq 0 \), we have \( |f'(h)| = 3|h|^2 \) (regardless of whether \( h \) is positive or negative), so
\[
\left| \frac{f'(h) - f'(0)}{h} \right| = \left| \frac{f'(h)}{h} \right| = \frac{|3|h|^2}{h} = 3|h|.
\]
Therefore
\[
f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = 0.
\]

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Finally, we consider $f^{(3)}(0)$. For $h > 0$, we have

$$\frac{f''(h) - f''(0)}{h} = \frac{6h}{h} = 6.$$ 

For $h < 0$, we have

$$\frac{f''(h) - f''(0)}{h} = \frac{-6h}{h} = -6.$$ 

So

$$f^{(3)}(0) = \lim_{h \to 0} \frac{f''(h) - f''(0)}{h}$$

does not exist. □

**Problem 5.22:** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that $x \in \mathbb{R}$ is a *fixed point* of $f$ if $f(x) = x$.

(a) Suppose that $f$ is differentiable and $f'(t) \neq 1$ for all $t \in \mathbb{R}$. Prove that $f$ has at most one fixed point.

*Solution.* Suppose $f$ has two distinct fixed points $r$ and $s$. Without loss of generality $r < s$. Apply the Mean Value Theorem on the interval $[r, s]$, to find $c \in (r, s)$ such that $f(s) - f(r) = f'(c)(s - r)$. Since $f(r) = r$ and $f(s) = s$, and since $s - r \neq 0$, this implies that $f'(c) = 1$. This contradicts the assumption that $f'(t) \neq 1$ for all $t \in \mathbb{R}$. □

(b) Define $f$ by $f(t) = t + (1 + e^t)^{-1}$ for $t \in \mathbb{R}$. Prove that $0 < f'(t) < 1$ for all $t \in \mathbb{R}$, but that $f$ has no fixed points. (You may use the standard properties of the exponential function from elementary calculus.)

*Solution.* If $x$ is a fixed point for $f$, then

$$x = f(x) = x + \frac{1}{1 + e^x},$$

whence

$$\frac{1}{1 + e^x} = 0.$$ 

This is obviously impossible.

Using the fact from elementary calculus that the derivative of $e^t$ is $e^t$, and using the differentiation rules proved in Chapter 5 of Rudin’s book, we get

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}.$$ 

Since $0 < e^t < 1 + e^t < (1 + e^t)^2$, we have

$$0 < \frac{e^t}{(1 + e^t)^2} < 1$$

for all $t$, from which it is clear that $0 < f'(t) < 1$ for all $t$. □

(c) Suppose there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all $t \in \mathbb{R}$. Prove that $f$ has a fixed point. Prove that if $x_0 \in \mathbb{R}$, and that if the sequence $(x_n)_{n \in \mathbb{Z}_{\geq 0}}$ is defined recursively by $x_{n+1} = f(x_n)$ for $n \in \mathbb{Z}_{\geq 0}$, then $(x_n)_{n \in \mathbb{Z}_{\geq 0}}$ converges to a fixed point of $f$. 
Solution. It suffices to prove the last statement. First, observe that the version of
the Mean Value Theorem in Theorem 5.19 of Rudin’s book implies that
$$|f(s) - f(t)| \leq A|s - t|$$
for all $s, t \in \mathbb{R}$.

Now let $x_0 \in \mathbb{R}$ be arbitrary, and define the sequence $(x_n)_{n \in \mathbb{Z}_{\geq 0}}$ recursively as in the statement. Using induction and the estimate above, we get
$$|x_{n+1} - x_n| \leq A^n|x_1 - x_0|$$
for all $n \in \mathbb{Z}_{\geq 0}$. Using the triangle inequality in the first step and the formula for the sum of a geometric series at the second step, we get, for $n, m \in \mathbb{Z}_{\geq 0}$,
$$|x_{n+m} - x_n| \leq \sum_{k=0}^{m-1} A^{n+k}|x_1 - x_0| = \frac{A^n - A^{n+m}}{1 - A} \cdot |x_1 - x_0| \leq \frac{A^n}{1 - A} \cdot |x_1 - x_0|.$$  

We now claim that $(x_n)_{n \in \mathbb{Z}_{\geq 0}}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $0 \leq A < 1$, we have $\lim_{N \to \infty} A^n = 0$, so we can choose $N$ so large that
$$\frac{A^N}{1 - A} \cdot |x_1 - x_0| < \varepsilon.$$  

For $m, n \geq N$, we then have
$$|x_m - x_n| \leq \frac{A^{\min(m, n)}}{1 - A} \cdot |x_1 - x_0| < \varepsilon.$$  

The claim is proved.

Since $\mathbb{R}$ is complete, $x = \lim_{n \to \infty} x_n$ exists. It is trivial that $\lim_{n \to \infty} x_{n+1} = x$ as well. Using the continuity of $f$ at $x$ in the first step, we get
$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x,$$
that is, $x$ is a fixed point for $f$. \qed

We can give an alternate proof of the existence of a fixed point, which is of interest, even though I do not see how to use it to show that the fixed point is the limit of the sequence described in the problem without essentially redoing the other solution.

Partial alternate solution. If $f(0) = 0$, we are done.

Next assume $f(0) > 0$. Define $b = f(0)(1 - A)^{-1} > 0$, and define $g(x) = f(x) - x$. Then $g$ is continuous and $g(0) > 0$. The version of the Mean Value Theorem in Theorem 5.19 of Rudin’s book implies that $|f(b) - f(0)| \leq Ab$. Therefore $f(b) \leq f(0) + Ab$, whence
$$g(b) = f(b) - b \leq f(0) + Ab - b = f(0) + (A - 1)f(0)(1 - A)^{-1} = 0.$$  

So the Intermediate Value Theorem provides $x \in [0, b]$ such that $g(x) = 0$, that is, $f(x) = x$.

The case $f(0) < 0$ follows by applying the case $f(0) > 0$ to the function $x \mapsto -f(-x)$. \qed

Problem 5.26: Let $f: [a, b] \to \mathbb{R}$ be differentiable, and suppose that $f(a) = 0$ and there is a real number $A$ such that $|f'(x)| \leq A|f(x)|$ for all $x \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. 


Hint: Fix $x_0 \in [a, b]$, and define
\[ M_0 = \sup_{x \in [a, b]} |f(x)| \quad \text{and} \quad M_1 = \sup_{x \in [a, b]} |f'(x)|. \]

For $x \in [a, b]$, we then have
\[ |f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0. \]

If $A(x_0 - a) < 1$, it follows that $M_0 = 0$. That is, $f(x) = 0$ for all $x \in [a, x_0]$. Proceed.

**Solution.** Choose $n \in \mathbb{Z}_{>0}$ and $x_0, x_1, \ldots, x_n \in [a, b]$ with
\[ a = x_0 < x_1 < \cdots < x_n = b \]
and such that $A(x_k - x_{k-1}) < 1$ for $k = 1, 2, \ldots, n$. We prove by induction on $k$ that $f(x) = 0$ for all $x \in [a, x_k]$. This is immediate for $k = 0$. So suppose it is known for some $k$; we prove it for $k + 1$.

Define
\[ M_0 = \sup_{x \in [x_{k-1}, x_k]} |f(x)| \quad \text{and} \quad M_1 = \sup_{x \in [x_{k-1}, x_k]} |f'(x)|. \]

The hypotheses imply that $M_1 \leq AM_0$. For $x \in [x_{k-1}, x_k]$, we have (using the induction hypothesis at the first step and the version of the Mean Value Theorem in Theorem 5.19 of Rudin’s book at the second step)
\[ |f(x)| = |f(x) - f(x_k)| \leq M_1(x - x_k) \leq AM_0(x - x_k). \]

In this inequality, take the supremum over all $x \in [x_{k-1}, x_k]$, getting
\[ M_0 = \sup_{x \in [x_{k-1}, x_k]} |f(x)| \leq AM_0 \sup_{x \in [x_{k-1}, x_k]} (x - x_k) = A(x_{k+1} - x_k)M_0. \]

Since $0 \leq A(x_{k+1} - x_k) < 1$ and $M_0 \geq 0$, this can only happen if $M_0 = 0$. Therefore $f(x) = 0$ for all $x \in [x_{k-1}, x_k]$, and hence for all $x \in [a, x_k]$. This completes the induction step, and the proof. □

**Problem 6.2:** Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous and nonnegative. Assume that $\int_a^b f = 0$. Prove that $f = 0$.

**Solution (sketch).** Assume that $f \neq 0$. Choose $x_0 \in [a, b]$ such that $f(x_0) > 0$. By continuity, there is $\delta > 0$ such that $f(x) > \frac{1}{2} f(x_0)$ for $|x - x_0| < \delta$. Let
\[ I = [a, b] \cap [x_0 - \frac{1}{2} \delta, x_0 + \frac{1}{2} \delta], \]
which is an interval of positive length, say $l$. Now construct a partition $P$ of $[a, b]$ such that $L(P, f) \geq l \cdot \frac{1}{2} f(x_0) > 0$. (This is easy but annoying to write down, since there are four cases: $a, b \not\in I$, $a, b \in I$, $a \in I$ but $b \not\in I$, and $a \not\in I$ but $b \in I$.) □

**Alternate solution (sketch).** Let $x_0$, $\delta$, $I$, and $l$ be as above. Let $\chi_I$ be the characteristic function of $I$. Check that $\chi_I$ is integrable and that $\int_a^b \chi_I = l$, by choosing for each $\varepsilon > 0$ a partition $P$ of $[a, b]$ such that $L(P, \chi_I) = l$ and $U(P, \chi_I) < l + \varepsilon$. (The same case breakdown as in the first solution will be needed.) Then $g = \frac{1}{2} f(x_0) \chi_I$ is integrable, and $\int_a^b g = \frac{1}{2} f(x_0) l$. Since $f \geq \chi_I$, we have $\int_a^b f \geq \int_a^b g > 0$. □
Problem 6.4: Let \(a, b \in \mathbb{R}\) with \(a < b\). Define \(f: [a, b] \rightarrow \mathbb{R}\) by
\[
f(x) = \begin{cases} 
0 & x \in \mathbb{R} \setminus \mathbb{Q} \\
1 & x \in \mathbb{Q}.
\end{cases}
\]
Prove that \(f\) is not Riemann integrable on \([a, b]\).

Solution (sketch). For every partition \(P = (x_0, x_1, \ldots, x_n)\) of \([a, b]\), every subinterval \([x_{j-1}, x_j]\) contains both rational and irrational numbers. Therefore \(L(P, f) = 0\) and \(U(P, f) = b - a\).

\[
\text{Problem A: Let } X \text{ be a metric space, and let } f: X \rightarrow X \text{ be a function. Suppose that there is a constant } k \text{ such that } d(f(x), f(y)) \leq kd(x, y) \text{ for all } x, y \in X.
\]

(1) Prove that \(f\) is uniformly continuous.
(2) Suppose that \(k < 1\) and \(X\) is complete. Prove that \(f\) has a unique fixed point, that is, there is a unique \(x \in X\) such that \(f(x) = x\).
(3) Show that the conclusion in Part (2) need not hold if \(X\) is not complete.

(I have rewritten the problem so that Part (1) does not assume completeness. This will not change the validity of anybody’s solution. The solution to (2) is very similar to the solution to Problem 5.22(c).)

Solution to (1) (sketch). For \(\varepsilon > 0\), take \(\delta = k^{-1}\varepsilon\).

Solution to (2). We claim that \(f\) has at most one fixed point. Let \(r, s \in X\) be fixed points. Then \(d(r, s) = d(f(r), f(s)) \leq kd(r, s)\). Since \(0 \leq k < 1\), this implies that \(d(r, s) = 0\), that is, \(r = s\). The claim is proved.

We now prove that \(f\) has at least one fixed point. Choose any \(x_0 \in X\). Define a sequence \((x_n)_{n \in \mathbb{Z}_{\geq 0}}\) recursively by \(x_{n+1} = f(x_n)\) for \(n \in \mathbb{Z}_{\geq 0}\). Using induction, we get
\[
d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)
\]
for all \(n \in \mathbb{Z}_{\geq 0}\). Using the triangle inequality in the first step and the formula for the sum of a geometric series at the second step, we get, for \(n \in \mathbb{Z}_{\geq 0}\) and \(m \in \mathbb{Z}_{>0}\),
\[
dx_{n+m}, x_n) \leq \sum_{k=0}^{m-1} k^{n+k}d(x_1, x_0) = \frac{k^n - k^{n+m}}{1 - k} \cdot d(x_1, x_0) \leq \frac{k^n}{1 - k} \cdot d(x_1, x_0).
\]

We now claim that \((x_n)_{n \in \mathbb{Z}_{\geq 0}}\) is a Cauchy sequence. Let \(\varepsilon > 0\). Since \(0 \leq k < 1\), we have \(\lim_{N \to \infty} k^N = 0\), so we can choose \(N\) so large that
\[
\frac{k^N}{1 - k} \cdot d(x_1, x_0) < \varepsilon.
\]
For \(m, n \geq N\), we then have
\[
d(x_m, x_n) \leq \frac{k^{\min(m, n)}}{1 - k} \cdot d(x_1, x_0) < \varepsilon.
\]
The claim is proved.

Since \(X\) is complete, \(x = \lim_{n \to \infty} x_n\) exists. It is trivial that \(\lim_{n \to \infty} x_{n+1} = x\) as well. Using the continuity of \(f\) at \(x\) in the first step, we get
\[
f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x,
\]
that is, \(x\) is a fixed point for \(f\). \(\Box\)
Solution to (3). Take $X = \mathbb{R} \setminus \{0\}$, with the restriction of the usual metric on $\mathbb{R}$, and define $f : X \to X$ by $f(x) = \frac{1}{2}x$ for all $x \in X$. Then $d(f(x), f(y)) = \frac{1}{2}d(x, y)$ for all $x, y \in X$, but clearly $f$ has no fixed point.

It follows from Part (2) that $X$ is not complete. (This is also easy to check directly: $(\frac{1}{n})_{n \in \mathbb{Z}_{>0} \setminus \{0\}}$ is a Cauchy sequence which does not converge.)

There are many other examples.