Generally, a “solution” is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A “solution (sketch)” is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A “solution (nearly complete)” is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

**Problem 7.22:** Let \( \alpha: [a, b] \to \mathbb{R} \) be nondecreasing, and let \( f: [a, b] \to \mathbb{C} \) be bounded. Assume that \( f \) is Riemann-Stieltjes integrable with respect to \( \alpha \) on \([a, b] \).

Prove that there is a sequence \((p_n)_{n \in \mathbb{Z}_{>0}}\) of polynomials such that
\[
\lim_{n \to \infty} \int_a^b |f - p_n|^2 \, d\alpha = 0.
\]
(Compare with Problem 6.12 of Rudin.)

**Solution (sketch).** In the notation of Problem 6.11 of Rudin (see an earlier solution set), the conclusion is that \( \lim_{n \to \infty} \|f - p_n\|^2_2 = 0 \). We prove the equivalent statement \( \lim_{n \to \infty} \|f - p_n\|_2 = 0 \).

Taking real and imaginary parts, without loss of generality \( f \) is real. (This reduction uses the triangle inequality for \( \|\cdot\|_2 \), Problem 6.11 of Rudin, which is solved in an earlier solution set.) Further, it is enough to find, for every \( \varepsilon > 0 \), a polynomial \( p \) such that \( \|f - p\|_2 < \varepsilon \).

Use Problem 6.12 of Rudin (solved in an earlier solution set) to find \( g \in C([0, 1], \mathbb{R}) \) such that \( \|f - g\|_2 < \frac{\varepsilon}{2} \). Use the Weierstrass theorem to find a polynomial \( p \) such that
\[
\|g - p\|_\infty < \frac{\varepsilon}{\sqrt{2(\alpha(b) - \alpha(a))}}.
\]
Then check that \( \|g - p\|_2 < \frac{\varepsilon}{2} \), so that triangle inequality for \( \|\cdot\|_2 \), Problem 6.11 of Rudin, implies that \( \|f - p\|_2 < \varepsilon \).

**Problem 7.24:** Instead of Rudin’s notation, for a metric space \( X \) we use the standard notation \( C_b(X) \) for the set of all bounded continuous functions from \( X \) to \( \mathbb{C} \).

Let \( X \) be a metric space with metric \( d \). Fix a point \( a \in X \). Assign to each \( p \in X \) the function \( f_p: X \to \mathbb{R} \) defined by
\[
f_p(x) = d(x, p) - d(x, a)
\]
for \( x \in X \). Prove that \( |f_p(x)| \leq d(a, p) \) for all \( x \in X \), and that \( f_p \in C_b(X) \). Prove that
\[
\|f_p - f_q\|_\infty = d(p, q)
\]
for all \( p, q \in X \).

*Date: 25 February 2019.*
If \( \Phi : X \to C_b(X) \) is defined by \( \Phi(p) = f_p \) for \( p \in X \), it follows that \( \Phi \) is an isometry (a distance-preserving mapping) from \( X \) to a subspace of \( C_b(X) \).

Conclusion: \( X \) is isometric to a dense subset of a complete metric space \( Y \).

**Solution.** To prove the first statement, let \( p \in X \). Then for all \( x \in X \) we have
\[
|f_p(x)| = |d(x, p) - d(x, a)| \leq d(a, p).
\]

To show that \( f_p \in C_b(X) \), it remains to prove continuity. Let \( x, y \in X \). We observe that
\[
|f_p(x) - f_p(y)| = |d(x, p) - d(x, a) - d(y, p) + d(y, a)|
\]
\[
\leq |d(x, p) - d(y, p)| + |d(x, a) - d(y, a)|
\]
\[
\leq 2d(x, y),
\]
that is, \( f_p \) is Lipshitz continuous with Lipshitz constant 2. It follows that \( f_p \in C_b(X) \).

Next we prove the third statement. Let \( p, q \in X \). Then
\[
|f_p(x) - f_q(x)| = |d(x, p) - d(x, a) - d(x, q) + d(x, a)|
\]
\[
= |d(x, p) - d(x, q)|
\]
\[
\leq d(p, q).
\]
It follows that \( \|f_p - f_q\|_{\infty} \leq d(p, q) \). Since
\[
|f_p(p) - f_q(p)| = |d(p, p) - d(p, a) - d(p, q) + d(p, a)| = d(p, q),
\]
we actually have
\[
\|f_p - f_q\|_{\infty} = d(p, q).
\]
Thus \( \Phi \) is an isometry. Hence \( X \) is isometric to \( \Phi(X) \). Let \( Y = \overline{\Phi(X)} \). Because \( C_b(X) \) is complete, \( Y \) is also complete. Thus \( X \) is isometric to a dense subset of a complete metric space. \( \square \)

**Problem 8.2:** For \( j, k \in \mathbb{Z}_{\geq 0} \), define \( a_{j,k} \in \mathbb{R} \) by
\[
a_{j,k} = \begin{cases} 
0 & j < k \\
-1 & j = k \\
\frac{1}{2^{k-j}} & j > k
\end{cases}
\]
That is, \( a_{j,k} \) is the number in the \( j \)-th row and \( k \)-th column of the array:
\[
\begin{array}{cccccc}
-1 & 0 & 0 & 0 & \ldots \\
\frac{1}{2} & -1 & 0 & 0 & \ldots \\
\frac{1}{4} & \frac{1}{2} & -1 & 0 & \ldots \\
\frac{1}{8} & \frac{1}{4} & \frac{1}{4} & -1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]
Prove that
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} = -2 \quad \text{and} \quad \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k} = 0.
\]
Solution (sketch). It is immediate from the formula for the sum of a geometric series that the column sums (which are clearly all the same) are all 0. Using the formula for the sum of a finite portion of a geometric series, one sees that the row sums are \(-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \ldots\), and the sum of these is \(-2\). □

Problem 8.3: Let \((a_{j,k})_{j,k \in \mathbb{Z}^+}\) be a doubly indexed family of nonnegative real numbers. Prove that
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k}.
\]
(The case \(+\infty = +\infty\) may occur.)

Comment. We give a solution which combines the finite and infinite cases. This is shorter than a solution using a case breakdown.

The result can also be gotten easily from Theorem 8.3 of Rudin’s book. See the alternate solution.

Solution. We show that
\[
(1) \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} = \sup \left\{ \sum_{(j,k) \in S} a_{j,k} : S \subset \mathbb{Z}^+ \times \mathbb{Z}^+ \text{ finite} \right\}.
\]
Since the right hand side is unchanged when \(j\) and \(k\) are interchanged, this will prove the result.

Let \(b\) be the right hand side of (1).

It is clear that if \(S \subset \mathbb{Z}^+ \times \mathbb{Z}^+\) is finite, then
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} \geq \sum_{(j,k) \in S} a_{j,k}.
\]
This implies
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} \geq b.
\]

For the reverse inequality, let \(s \in \mathbb{R}\) be an arbitrary number satisfying
\[
s < \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k};
\]
we prove that \(b > s\). Choose \(\varepsilon > 0\) such that
\[
s + \varepsilon < \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}.
\]
Choose \(m \in \mathbb{N}\) such that
\[
\sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{j,k} > s + \frac{\varepsilon}{2}.
\]
If for some \(j_0 \in \{1, 2, \ldots, m\}\), the sum \(\sum_{k=1}^{\infty} a_{j_0,k}\) is infinite, choose \(n\) such that
\[
\sum_{k=1}^{n} a_{j_0,k} > s.\]
Then clearly \(S = \{j_0\} \times \{1, 2, \ldots, n\}\) is a finite subset of \(\mathbb{Z}^+ \times \mathbb{Z}^+\).
such that $\sum_{(j,k) \in S} a_{j,k} > s$, and we are done. Otherwise, for $1 \leq j \leq m$ choose $n_j \in \mathbb{Z}_{> 0}$ such that

$$\sum_{k=1}^{n_j} a_{j,k} > \sum_{k=1}^{\infty} a_{j,k} - \frac{2m}{2m}.$$

Set

$$n = \max(n_1, n_2, \ldots, n_m) \quad \text{and} \quad S = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}.$$

Then

$$\sum_{(j,k) \in S} a_{j,k} > \sum_{j=1}^{m} \left( \sum_{k=1}^{\infty} a_{j,k} - \frac{\varepsilon}{2m} \right) = \sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{j,k} - \frac{\varepsilon}{2} > s.$$

This proves the desired inequality. \(\square\)

**Note.** We can even avoid the case breakdown in the last paragraph, as follows. Choose $b_1, b_2, \ldots, b_m$ such that $b_1 + b_2 + \cdots + b_m > s$ and $b_j < \sum_{k=1}^{\infty} a_{j,k}$ for $1 \leq j \leq m$. Then choose $n_j \in \mathbb{Z}_{> 0}$ such that $\sum_{k=1}^{n_j} a_{j,k} > b_j$. However, at this point the case breakdown seems easier.

**Alternate Solution.** First suppose that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} < \infty$. Since the terms $a_{j,k}$ are nonnegative, Theorem 8.3 of Rudin’s book shows that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k}$.

Next, suppose that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} < \infty$. We then apply Theorem 8.3 of Rudin’s book with $c_{j,k} = a_{k,j}$ (reversing the order of the indices) in place of $a_{j,k}$, to get $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}$.

Finally, if neither of the above cases holds, then the sums $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}$ and $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k}$ are both infinite, and hence equal. \(\square\)

**Problem 8.4:** Prove the following limit relations.

(a) $\lim_{x \to 0} \frac{b^x - 1}{x} = \log(b)$ for $b > 0$.

**Solution (sketch).** Set $f(x) = b^x = \exp(x \log(b))$. The desired limit is by definition $f'(0)$, which can be gotten from the second expression for $f$ using the chain rule. \(\square\)

(b) $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$.

**Solution (sketch).** This limit is $f'(0)$ with $f(x) = \log(1 + x)$.

(c) $\lim_{x \to 0} (1 + x)^{1/x} = e$.

**Solution (sketch).** By Part (b), we have

$$\lim_{x \to 0} \log \left( (1 + x)^{1/x} \right) = 1.$$

Apply exp to both sides, using continuity of exp. \(\square\)

(d) $\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x$. 
Solution (sketch). Write
\[
(1 + \frac{x}{n})^n = \left(1 + \frac{x}{n}\right)^{n/x}
\]
ote that \(\frac{x}{n} \to 0\) as \(n \to \infty\), and apply Part (c) (using continuity at the appropriate places).

\[\square\]

Problem 8.5: Find the following limits.

(a) \(\lim_{x \to 0} \frac{e - (1 + x)^{1/x}}{x}\).

Comment. We first do a slightly heuristic calculation, which shows what the answer is, and then give a sketch of a solution in the correct logical order.

Calculation (sketch). Rewrite and then use L'Hospital’s Rule to get
\[
\lim_{x \to 0} \frac{e - (1 + x)^{1/x}}{x} = \lim_{x \to 0} \frac{e - \exp\left(\frac{1}{x} \log(1 + x)\right)}{x} = \lim_{x \to 0} \frac{(x + 1)^{1/x} - \log(x + 1)}{x^2}.
\]

Rewrite:
\[
-\lim_{x \to 0} \frac{(x + 1)^{1/x}}{x^2} \left(\frac{1}{x(x + 1)} - \frac{\log(x + 1)}{x^2}\right)
= -\lim_{x \to 0} \frac{(x + 1)^{1/x}}{x^2} \left(\lim_{x \to 0} \frac{\log(x + 1)}{x^2}\right)
= -e \lim_{x \to 0} \frac{x}{x + 1} - \log(x + 1).
\]

Use L’Hospital’s rule to get
\[
-e \lim_{x \to 0} \frac{x + 1}{x^2} - \log(x + 1) = e \lim_{x \to 0} \frac{1}{x + 1} - \frac{1}{x(x + 1)^2} = e \lim_{x \to 0} \frac{1}{2(x + 1)^2} = e/2.
\]

Solution (sketch). We observe that
\[
\lim_{x \to 0} \frac{1}{x + 1} - \frac{1}{x(x + 1)^2} = \lim_{x \to 0} \frac{1}{2(x + 1)^2}
\]
exists (and is equal to \(\frac{1}{2}\)). Since the other hypotheses of L’Hospital’s Rule are also satisfied, we can use it to show that
\[
-\lim_{x \to 0} \frac{x + 1}{x^2} - \log(x + 1) = \lim_{x \to 0} \frac{1}{x + 1} - \frac{1}{x(x + 1)^2} = \frac{1}{2}.
\]

Therefore
\[
-\lim_{x \to 0} (x + 1)^{1/x} \left(\frac{1}{x(x + 1)} - \frac{\log(x + 1)}{x^2}\right) = e/2.
\]

Since the other hypotheses of L’Hospital’s Rule are also satisfied, we can use it again to show that
\[
\lim_{x \to 0} \frac{e - (1 + x)^{1/x}}{x} = \lim_{x \to 0} (x + 1)^{1/x} \left(\frac{1}{x(x + 1)} - \frac{\log(x + 1)}{x^2}\right) = e/2.
\]
This completes the solution. □

(b) \( \lim_{n \to \infty} \frac{n}{\log(n)} (n^{1/n} - 1) \).

**Solution.** Rewrite the limit as

\[
\lim_{n \to \infty} \frac{n}{\log(n)} (n^{1/n} - 1) = \lim_{n \to \infty} \frac{\exp \left( \frac{1}{n} \log(n) \right) - 1}{\frac{1}{n} \log(n)}
\]

Now \( \lim_{n \to \infty} \frac{1}{n} \log(n) = 0 \), so

\[
\lim_{n \to \infty} \exp \left( \frac{1}{n} \log(n) \right) - 1 = \lim_{h \to 0} \frac{\exp(h) - 1}{h} = \exp'(0) = 1.
\]

This completes the solution. □

(c) \( \lim_{x \to 0} \frac{\tan(x) - x}{x[1 - \cos(x)]} \).

**Solution 1 (sketch).** Write

\[
\frac{\tan(x) - x}{x[1 - \cos(x)]} = \left( \frac{1}{\cos(x)} \right) \left( \frac{\sin(x) - x \cos(x)}{x - x \cos(x)} \right).
\]

Since \( \lim_{x \to 0} \cos(x) = 1 \), we get

\[
\lim_{x \to 0} \frac{\tan(x) - x}{x[1 - \cos(x)]} = \lim_{x \to 0} \frac{\sin(x) - x \cos(x)}{x - x \cos(x)},
\]

provided the limit on the right exists. Now apply L'Hospital's rule three times (being sure to check that the hypotheses are satisfied!):

\[
\lim_{x \to 0} \frac{\sin(x) - x \cos(x)}{x - x \cos(x)} = \lim_{x \to 0} \frac{x \sin(x)}{x \cos(x) + \sin(x)} = \lim_{x \to 0} \frac{2 \cos(x) - x \sin(x)}{3 \cos(x) - x \sin(x)} = \frac{2}{3}.
\]

(See the solution to Part (a) for the logically correct way to write this.) □

**Solution 2 (sketch; not recommended).** Apply L'Hospital's rule three times to the original expression (being sure to check that the hypotheses are satisfied!):

\[
\lim_{x \to 0} \frac{\tan(x) - x}{x[1 - \cos(x)]} = \lim_{x \to 0} \frac{\sec(x)^2 - 1}{1 - \cos(x) + x \sin(x)} = \lim_{x \to 0} \frac{2[\sec(x)]^2 \tan(x)}{x \cos(x) + 2 \sin(x)} = \lim_{x \to 0} \frac{2[\sec(x)]^4 + 4[\sec(x)]^2 [\tan(x)]^2}{3 \cos(x) - x \sin(x)} = \frac{2}{3}.
\]

(See the solution to Part (a) for the logically correct way to write this.)

This solution is not recommended because of the messiness of the differentiation. (The results given here were obtained using Mathematica.) □

**Solution 3 (sketch).** The mathematical justification for the power series manipulations is easy but is not provided here. We compute the limit by substitution of power series. We check easily that the usual power series

\[
\sin(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots \quad \text{and} \quad \cos(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots
\]
follow from the definitions of \( \sin(x) \) and \( \cos(x) \) in terms of \( \exp(ix) \) and from the definition of \( \exp(z) \). Start from the equivalent limit given in Solution 1. We have

\[
\frac{\sin(x) - x \cos(x)}{x - x \cos(x)} = \frac{\left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots\right) - x \left(1 - \frac{1}{3!} x^2 + \frac{1}{5!} x^4 - \cdots\right)}{x - \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots\right)}
\]

\[
= \frac{\left(\frac{1}{3!} - \frac{1}{5!}\right) x^3 + \left(\frac{1}{3!} - \frac{1}{5!}\right) x^5 + \cdots}{\frac{1}{3!} x^3 - \frac{1}{5!} x^5 + \cdots}
\]

\[
= \frac{\left(\frac{1}{3!} - \frac{1}{5!}\right) + \left(\frac{1}{3!} - \frac{1}{5!}\right) x^2 + \cdots}{\frac{1}{3!} - \frac{1}{5!} x^2 + \cdots}.
\]

The last expression defines a function of \( x \) which is continuous at 0, so

\[
\lim_{x \to 0} \frac{\tan(x) - x}{x(1 - \cos(x))} = \lim_{x \to 0} \frac{\sin(x) - x \cos(x)}{x - x \cos(x)} = \frac{\left(\frac{1}{3!} - \frac{1}{5!}\right)}{\left(\frac{1}{3!}\right)} = \frac{2}{3}.
\]

This completes the solution. □

(d) \( \lim_{x \to 0} \frac{x - \sin(x)}{\tan(x) - x} \).

Solution 1 (sketch). Write

\[
\frac{x - \sin(x)}{\tan(x) - x} = \cos(x) \left(\frac{x - \sin(x)}{\sin(x) - x \cos(x)}\right).
\]

Since \( \lim_{x \to 0} \cos(x) = 1 \), we get

\[
\lim_{x \to 0} \frac{x - \sin(x)}{\tan(x) - x} = \lim_{x \to 0} \frac{x - \sin(x)}{\sin(x) - x \cos(x)}
\]

provided the limit on the right exists. Now apply L’Hospital’s rule three times (being sure to check that the hypotheses are satisfied!):

\[
\lim_{x \to 0} \frac{x - \sin(x)}{\sin(x) - x \cos(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)} = \lim_{x \to 0} \frac{x}{x \cos(x) + \sin(x)}
\]

\[
= \lim_{x \to 0} \frac{\cos(x)}{2 \cos(x) - x \sin(x)} = \frac{1}{2}.
\]

(See the solution to Part (a) for the logically correct way to write this.) □

Solution 2 (sketch). Apply L’Hospital’s rule twice to the original expression (being sure to check that the hypotheses are satisfied!), and cancelling the common factor \( \sin(x) \) in the third step:

\[
\lim_{x \to 0} \frac{x - \sin(x)}{\tan(x) - x} = \lim_{x \to 0} \frac{1 - \cos(x)}{\frac{\sec(x)}{2[\sec(x)]^2 - 1}} = \lim_{x \to 0} \frac{\sin(x)}{2[\sec(x)]^2 \tan(x)} = \lim_{x \to 0} \frac{1}{2[\sec(x)]^3} = \frac{1}{2}.
\]

(See the solution to Part (a) for the logically correct way to write this.) □

Solution 3 (sketch). Use the same method as Solution 3 to Part (c). Details are omitted. □

Solution 4: L’Hospital’s rule shows that, if the limit on the right exists, then

\[
\lim_{x \to 0} \frac{x - \sin(x)}{\tan(x) - x} = \lim_{x \to 0} \frac{1 - \cos(x)}{\frac{\sec(x)}{2[\sec(x)]^2 - 1}}.
\]
To calculate the limit on the right, we multiply the numerator and denominator by $[\cos(x)]^2$, factor the denominator, and then cancel common factors:

$$\frac{1 - \cos(x)}{[\sec(x)]^2 - 1} = \frac{[1 - \cos(x)][\cos(x)]^2}{1 - [\cos(x)]^2} = \frac{[1 - \cos(x)][\cos(x)]^2}{[1 - \cos(x)][1 + \cos(x)]} = \frac{[\cos(x)]^2}{1 + \cos(x)}.$$ 

So

$$\lim_{x \to 0} \frac{1 - \cos(x)}{[\sec(x)]^2 - 1} = \lim_{x \to 0} \frac{[\cos(x)]^2}{1 + \cos(x)} = \frac{1}{2}.$$ 

whence

$$\lim_{x \to 0} \frac{x - \sin(x)}{\tan(x) - x} = \lim_{x \to 0} \frac{1 - \cos(x)}{[\sec(x)]^2 - 1} = \frac{1}{2}.$$ 

This completes the solution.