Generally, a “solution” is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A “solution (sketch)” is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A “solution (nearly complete)” is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 8.6: Let \( f : \mathbb{R} \to \mathbb{R} \) be a nonzero function satisfying \( f(x + y) = f(x)f(y) \) for all \( x, y \in \mathbb{R} \).

(a) Suppose \( f \) is differentiable. Prove that there is \( c \in \mathbb{R} \) such that \( f(x) = \exp(cx) \) for all \( x \in \mathbb{R} \).

Solution. Since \( f(0)f(x) = f(x) \) for all \( x \), and since there is some \( x \in \mathbb{R} \) such that \( f(x) \neq 0 \), it follows that \( f(0) = 1 \). The computation

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f(x)f'(0)
\]

shows that

\[
f'(x) = f(x)f'(0)
\]

for all \( x \in \mathbb{R} \). Now let \( g(x) = f(x) \exp(-xf'(0)) \) for \( x \in \mathbb{R} \). Using the product rule and (1), for \( x \in \mathbb{R} \) we get

\[
g'(x) = f'(x) \exp(-xf'(0)) - f(x)f'(0) \exp(-xf'(0)) = 0.
\]

So \( g \) is constant. Since \( f(0) = 1 \), we get \( g(0) = 1 \). Therefore \( f(x) = \exp(xf'(0)) \) for all \( x \in \mathbb{R} \).

Alternate solution. Use the solution to Part (b).

(b) Suppose \( f \) is continuous. Prove that there is \( c \in \mathbb{R} \) such that \( f(x) = \exp(cx) \) for all \( x \in \mathbb{R} \).

Solution. Since \( f(0)f(x) = f(x) \) for all \( x \), and since there is some \( x \in \mathbb{R} \) such that \( f(x) \neq 0 \), it follows that \( f(0) = 1 \). By continuity of \( f \) at 0, there is \( a > 0 \) such that \( f(a) > 0 \). Define

\[
g(x) = f(x) \exp(-a^{-1}x \log(f(a)))
\]

for \( x \in \mathbb{R} \). Then \( g \) is continuous and satisfies \( g(x + y) = g(x)g(y) \) for all \( x, y \in \mathbb{R} \), and \( g(a) = 1 \). Set

\[
S = \{ x > 0 : g(x) = 1 \} \quad \text{and} \quad x_0 = \inf(S).
\]

We have \( S \neq \emptyset \) because \( a \in S \), so \( x_0 \in [0, \infty) \).

We claim that \( x_0 = 0 \). Suppose not. We have \( g(x_0) = 1 \) by continuity. Moreover,

\[
g\left(\frac{1}{2}x_0\right)^2 = g(x_0) = 1 \text{ but } g\left(\frac{1}{4}x_0\right) \neq 1, \text{ so } g\left(\frac{1}{2}x_0\right) = -1. \text{ Then } g\left(\frac{1}{4}x_0\right)^2 =
\]

Date: 4 March 2019.
\[ g\left(\frac{1}{2}x_0\right) = -1, \quad \text{contradicting the assumption that } g\left(\frac{1}{2}x_0\right) \text{ is real. This proves the claim.} \]

We claim that \( g(x) = 1 \) for all \( x \in \mathbb{R} \). Let \( x \in \mathbb{R} \) and let \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that whenever \( y \in \mathbb{R} \) satisfies \( |y - x| < \delta \), then \( |g(y) - g(x)| < \varepsilon \). By the definition of \( S \) and because \( x_0 = 0 \), there is \( z \in S \) such that \( 0 < z < \delta \). Choose \( n \in \mathbb{Z} \) such that \( |nz - x| < \delta \). Then \( g(nz) = g(z)^n = 1 \), so \( |g(x) - 1| < \varepsilon \). This proves the claim.

Set \( c = a^{-1}\log(f(a)) \). We have shown that \( f(x) = \exp(cx) \) for all \( x \in \mathbb{R} \). \( \square \)

Alternate solution. We have \( f(0) = 1 \) for the same reason as in the first solution. Furthermore, for \( x \in \mathbb{R} \) we have \( f\left(\frac{1}{2}x\right) \in \mathbb{R} \), so

\[
 f(x) = f\left(\frac{1}{2}x\right)^2 \geq 0.
\]

We claim that \( f(x) > 0 \) for all \( x \in \mathbb{R} \). To prove the claim, suppose \( x \in \mathbb{R} \) and \( f(x) = 0 \). Then \( f\left(\frac{1}{2}x\right) = 0 \), and by induction \( f(2^{-n}x) = 0 \) for all \( n \in \mathbb{Z}_{>0} \). Since \( f \) is continuous and \( \lim_{n \to \infty} 2^{-n}x = 0 \), this contradicts \( f(0) = 1 \). The claim is proved.

Define \( g(x) = \exp(x \log(f(1))) \) for \( x \in \mathbb{R} \). For \( n \in \mathbb{Z}_{>0} \),

\[
 f\left(\frac{1}{n}\right)^n = f(1) \quad \text{and} \quad g\left(\frac{1}{n}\right)^n = g(1) = f(1).
\]

Since both \( f\left(\frac{1}{n}\right) \) and \( g\left(\frac{1}{n}\right) \) are positive, and positive \( n \)-th roots are unique, it follows that \( f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \) for all \( n \in \mathbb{Z}_{>0} \).

We claim that \( f(x) = g(x) \) for all \( x \in \mathbb{Q} \). We prove the claim. It is certainly true for \( x = 0 \). Next, assume that \( x \in \mathbb{Q} \) and \( x > 0 \). Then there are \( m, n \in \mathbb{Z}_{>0} \) such that \( x = \frac{m}{n} \). By induction on \( m \), \( f(my) = f(y)^m \) for all \( y \in \mathbb{R} \). So

\[
 f\left(\frac{m}{n}\right) = f\left(\frac{1}{n}\right)^m = g\left(\frac{1}{n}\right)^m = g\left(\frac{m}{n}\right).
\]

Finally, assume \( x < 0 \). Since \( f(0) = 1 \), we have \( f(x) = f(-x)^{-1} \), so, by the case just done,

\[
 f(x) = f(-x)^{-1} = g(-x)^{-1} = g(x).
\]

The claim is proved.

Since \( f \) and \( g \) are continuous, and since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), it follows that \( f = g \). Set \( c = \log(f(1)) \). We have shown that \( f(x) = \exp(cx) \) for all \( x \in \mathbb{R} \). \( \square \)

**Problem 8.7:** Prove that

\[
 \frac{2}{\pi} < \frac{\sin(x)}{x} < 1
\]

for \( 0 < x < \frac{\pi}{2} \).

**Solution (sketch).** The inequality \( \frac{\sin(x)}{x} < 1 \) is the same as \( \sin(x) < x \). This is proved by noting that \( \sin(0) = 0 \) and that the derivative \( 1 - \cos(x) \) of \( x - \sin(x) \) is strictly positive for \( 0 < x < \frac{\pi}{2} \).

This also implies that \( \frac{\sin(x)}{x} < 1 \) for \( x = \frac{\pi}{2} \), so that \( \frac{2}{\pi} < 1 \).

For the other inequality, suppose there is \( x_0 \) with \( 0 < x_0 < \frac{\pi}{2} \) such that

\[
 \frac{2}{\pi} \geq \frac{\sin(x_0)}{x_0}.
\]
Using the Intermediate Value Theorem and \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \), there is \( x_0 \) with 
\[ 0 < x_0 < \frac{\pi}{2} \]
such that 
\[ \frac{2}{\pi} = \frac{\sin(x_0)}{x_0} \].
Set \( f(x) = \sin(x) - \frac{2}{\pi} \cdot x \). For \( 0 < x \leq \frac{\pi}{2} \), we have \( f''(x) = -\sin(x) < 0 \), so that \( f' \) is strictly decreasing on \([0, \frac{\pi}{2}]\). The Mean Value Theorem gives \( z \in (0, x_0) \) such that 
\[ f'(z) = 0, \] so \( f'(x) < 0 \) for \( x_0 \leq x \leq \frac{\pi}{2} \). Since \( f(x_0) = 0 \), we get \( f\left(\frac{\pi}{2}\right) < 0 \), a contradiction. \( \square \)

Alternate Solution (sketch). Suppose \( f: [0, a] \to \mathbb{R} \) is a continuous function such that:

1. \( f(0) = 0 \).
2. \( f'(x) \) exists for \( x \in (0, a) \).
3. \( f' \) is strictly decreasing on \((0, a)\).

We claim that the function \( g(x) = x^{-1}f(x) \) is strictly decreasing on \((0, a)\).

To see that the claim implies the result, take \( f(x) = \sin(x) \) and \( a = \frac{\pi}{2} \). We conclude that \( g \) is strictly decreasing on \((0, \frac{\pi}{2})\). We get the result by observing that
\[ \lim_{x \to 0^+} \frac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1. \]

To prove the claim, let \( x_1, x_2 \in \mathbb{R} \) satisfy \( 0 < x_1 < x_2 \leq b \). Use the Mean Value Theorem to choose \( c_1 \in (0, x_1) \) and \( c_2 \in (x_1, x_2) \) such that
\[ \frac{f(x_1)}{x_1} - \frac{f(0)}{x_1 - 0} = f'(c_1) \quad \text{and} \quad \frac{f(x_2)}{x_2} - \frac{f(x_1)}{x_2 - x_1} = f'(c_2). \]
Then \( c_1 < c_2 \), so \( f'(c_1) > f'(c_2) \), whence
\[ \frac{f(x_1)}{x_1} > \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \]
Multiply both sides of this inequality by \( x_1(x_2 - x_1) > 0 \) to get
\[ (x_2 - x_1)f(x_1) > x_1(f(x_2) - f(x_1)). \]
Multiply out and cancel \(-x_1f(x_1)\) to get \( x_2f(x_1) > x_1f(x_2) \), and divide by \( x_1x_2 \) to get \( g(x_1) > g(x_2) \). \( \square \)

Alternate Solution 2 (sketch). Since
\[ \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi} \quad \text{and} \quad \lim_{x \to 0^+} \frac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1, \]
it suffices to prove that the function \( g(x) = x^{-1}\sin(x) \) is strictly decreasing on \((0, \frac{\pi}{2})\). We do this by showing that \( g'(x) < 0 \) on this interval.

Begin by observing that the function \( h(x) = x\cos(x) - \sin(x) \) satisfies \( h(0) = 0 \) and \( h'(x) = -\sin(x) \) for all \( x \). Therefore \( h'(x) < 0 \) for \( x \in (0, \frac{\pi}{2}) \), and from \( h(0) = 0 \) we get \( h(x) < 0 \) for \( x \in (0, \frac{\pi}{2}) \). Now calculate, for \( x \in (0, \frac{\pi}{2}) \),
\[ g'(x) = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{h(x)}{x^2} < 0. \]
This is the required estimate. \( \square \)
Alternate Solution 3 (sketch). As in the second alternate solution, we set \( g(x) = x^{-1} \sin(x) \) and show that \( g'(x) < 0 \) for \( x \in \left(0, \frac{\pi}{2}\right) \). Define \( q(x) = \frac{\sin(x)}{\cos(x)} - x \) for \( x \in \left[0, \frac{\pi}{2}\right) \). Then the quotient rule and the relation \( \sin^2(x) + \cos^2(x) = 1 \) give \( q'(x) = \frac{1}{\cos^2(x)} - 1 \).

For \( x \in \left(0, \frac{\pi}{2}\right) \) we have \( 0 < \cos(x) < 1 \), from which it follows that \( q'(x) > 0 \). Since \( q \) is continuous on \( \left[0, \frac{\pi}{2}\right) \) and \( q(0) = 0 \), we get \( q(x) > 0 \) for \( x \in \left(0, \frac{\pi}{2}\right) \). It follows that \( \sin(x) > x \cos(x) \) for \( x \in \left(0, \frac{\pi}{2}\right) \). As in the second alternate solution, this implies that \( g'(x) < 0 \) for \( x \in \left(0, \frac{\pi}{2}\right) \). \( \square \)

Problem 8.8: For \( n \in \mathbb{Z}_{\geq 0} \) and \( x \in \mathbb{R} \), prove that
\[
|\sin(nx)| \leq n|\sin(x)|.
\]
Note that this inequality may be false when \( n \) is not an integer. For example,
\[
|\sin\left(\frac{1}{2}\right)| > \frac{1}{2}|\sin(\pi)|.
\]
Solution (sketch). Combining the formula \( \exp(i(x + y)) = \exp(ix) \exp(iy) \) with either the result
\[
\cos(x) = \text{Re}(\exp(ix)) \quad \text{and} \quad \sin(x) = \text{Im}(\exp(ix))
\]
(for \( x \) real) or the definitions
\[
\cos(x) = \frac{1}{2} [\exp(ix) + \exp(-ix)] \quad \text{and} \quad \sin(x) = \frac{1}{2i} [\exp(ix) - \exp(-ix)],
\]
prove the addition formula
\[
\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)
\]
for all \( x \) and \( y \). (Using the first suggestion gives this only for real \( x \) and \( y \), but that is all that is needed here.)

Now prove the result by induction on \( n \). For \( n = 0 \) the desired inequality says \( 0 \leq 0 \) for all \( x \), which is certainly true. Assuming it is true for \( n \), we have (using the addition formula in the first step and the induction hypothesis and the inequality \( |\cos(a)| \leq 1 \) for all real \( a \) in the second step)
\[
|\sin((n+1)x)| \leq |\sin(x)| \cdot |\cos(nx)| + |\cos(x)| \cdot |\sin(nx)|
\]
\[
\leq |\sin(x)| + n|\sin(x)| = (n+1)|\sin(x)|
\]
for all \( x \in \mathbb{R} \). \( \square \)

Alternate Solution (sketch). We first prove the inequality for \( 0 \leq x \leq \frac{\pi}{2} \). If \( x \in \left[0, \frac{\pi}{2}\right] \) satisfies \( \sin(x) \geq \frac{1}{n} \), then since \( |\sin(nx)| \leq 1 \) there is nothing to prove. Otherwise, \( x < \frac{\pi}{2n} \) by Problem 8.7. Since \( t \mapsto \cos(t) \) is nonincreasing on \( \left[0, \frac{\pi}{2}\right] \) (its derivative \( -\sin(t) \) is nonpositive there), we get \( \cos(nx) \leq \cos(x) \) for \( x \in \left[0, \frac{\pi}{2n}\right] \). With \( f(x) = n \sin(x) - \sin(nx) \), we therefore have \( f(0) = 0 \) and
\[
f'(x) = n \left[\cos(x) - \cos(nx)\right] \leq 0
\]
for \( x \in [0, \frac{\pi}{2n}] \). Since also \( \sin(nx) \geq 0 \) on this interval, the inequality is proved for 
\( 0 < x \leq \frac{\pi}{2n} \) and hence \( 0 \leq x \leq \frac{\pi}{2} \).

For \( -\frac{\pi}{2} \leq x < 0 \), the inequality follows from the fact that \( t \mapsto \sin(t) \) is an
odd function. (This is easily seen from the definition.) For \( \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \), reduce
to \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \) using the identity \( \sin(k(x + \pi)) = (-1)^k \sin(kx) \) (which is easily
derived from the definition and \( \exp(i\pi) = -1 \)). The inequality now follows for all
\( x \) by periodicity. \( \square \)

Problem 8.9: (a) For \( n \in \mathbb{Z}_{>0} \) set
\[
\gamma = \lim_{n \to \infty} [s_n - \log(n)]
\]
exists. (This limit is called Euler’s constant. Numerically, \( \gamma \approx 0.5772 \). It is not
known whether \( \gamma \) is rational or not.)

Solution. We have
\[
\frac{1}{n} - [\log(n+1) - \log(n)] = \frac{1}{n} - \int_n^{n+1} \frac{1}{t} \, dt = \int_n^{n+1} \left( \frac{1}{n} - \frac{1}{t} \right) \, dt.
\]
for \( n \in \mathbb{Z}_{>0} \). Since the integrand is between 0 and \( \frac{1}{n} - \frac{1}{n+1} \), it follows that
\[
0 \leq \frac{1}{n} - [\log(n+1) - \log(n)] \leq \frac{1}{n} - \frac{1}{n+1}.
\]
Since
\[
\sum_{k=1}^{n} \left( \frac{1}{k} - [\log(k+1) - \log(k)] \right) = s_n - \log(n+1),
\]
we get by adding up terms
\[
s_n - \log(n+1) \leq 1 - \frac{1}{n+1} < 1
\]
for all \( n \in \mathbb{Z}_{>0} \); also,
\[
s_n - \log(n+1) = s_{n-1} - \log(n) + \frac{1}{n} - [\log(n+1) - \log(n)] \geq s_{n-1} - \log(n).
\]
Therefore \((s_n - \log(n+1))_{n \in \mathbb{Z}_{>0}}\) is a bounded nondecreasing sequence, hence converges.

Next observe that
\[
\log(n+1) - \log(n) = \int_n^{n+1} \frac{1}{t} \, dt,
\]
so that
\[
0 \leq \log(n+1) - \log(n) \leq \frac{1}{n}.
\]
It follows that \( \lim_{n \to \infty} [\log(n+1) - \log(n)] = 0 \), whence
\[
\lim_{n \to \infty} [s_n - \log(n)] = \lim_{n \to \infty} [s_n - \log(n+1)].
\]
This completes the solution. \( \square \)
Alternate Solution 1. Let $P_n$ be the partition $P_n = (1, 2, \ldots, n)$ of $[1, n]$. Set $f(x) = x^{-1}$ for $x \in (0, \infty)$. Since $f$ is nonincreasing, we have

$$L(P_n, f) = \sum_{k=2}^{n} \frac{1}{k} \quad \text{and} \quad U(P_n, f) = \sum_{k=1}^{n-1} \frac{1}{k}.$$  

Therefore

$$s_n - \log(n) = 1 + L(P_n, f) - \int_{1}^{n} f = 1 + \frac{1}{n} + U(P_n, f) - \int_{1}^{n} f.$$  

We claim that $(s_n - \log(n))_{n \in \mathbb{Z}_{>0}}$ is nonincreasing. To prove the claim, let $n \in \mathbb{Z}_{>0}$. Then

$$[s_n - \log(n)] - [s_{n+1} - \log(n+1)] = \left[1 + L(P_n, f) - \int_{1}^{n} f\right] - \left[1 + L(P_{n+1}, f) - \int_{1}^{n+1} f\right]$$

$$= \int_{n}^{n+1} f - [L(P_{n+1}, f) - L(P_n, f)] = \int_{n}^{n+1} f - \frac{1}{n+1}. $$

The last expression is nonnegative because $f(x) \geq \frac{1}{n+1}$ for all $x \in [n, n+1]$. The claim is proved.

We claim that $(s_n - \log(n))_{n \in \mathbb{Z}_{>0}}$ is nonincreasing and bounded below, $\lim_{n \to \infty} (s_n - \log(n))$ exists. \qed

Alternate Solution 2. First prove the following lemma.

**Lemma:** For every $x \geq 0$, we have $0 \leq x - \log(1 + x) \leq \frac{1}{2}x^2$.

**Proof.** Set $g(x) = x - \log(1 + x)$ and $h(x) = \frac{1}{2}x^2$ for $x \in (-1, \infty)$. Then $g(0) = h(0) = 0$. Furthermore, for all $x \geq 0$ we have

$$g'(x) = 1 - \frac{1}{1 + x} = \frac{x}{1 + x} \geq 0 \quad \text{and} \quad h'(x) - g'(x) = x - \frac{x}{1 + x} = \frac{x^2}{1 + x} \geq 0.$$  

Therefore $0 \leq g(x) \leq h(x)$ for all $x \geq 0$. \qed

Now define

$$b_k = \frac{1}{k} - \left[\log(k) - \log(k + 1)\right] = \frac{1}{k} - \log\left(1 + \frac{1}{k}\right)$$

for $k \in \mathbb{Z}_{>0}$. (That the two expressions are equal follows from the algebraic properties of the function log. See Equation (40) on Page 181 of Rudin’s book.) Since $\log(1) = 0$, we have

$$s_n - \log(n + 1) = \sum_{k=1}^{n} b_k.$$  

By the lemma, we have $0 \leq b_k \leq \frac{1}{2}k^{-2}$ for $k \geq 1$. Since $\sum_{k=1}^{\infty} k^{-2}$ converges, the Comparison Test shows that $\sum_{k=1}^{\infty} b_k$ converges, whence $\lim_{n \to \infty} [s_n - \log(n + 1)]$ exists.
Now show that \( \lim_{n \to \infty} [\log(n + 1) - \log(n)] = 0 \) as in the first solution, and conclude as there that \( \lim_{n \to \infty} [s_n - \log(n)] \) exists.

Alternate Solution 3 (outline). This solution differs from the previous one only in the method of proof of the lemma. Instead of comparing derivatives, we use the derivative form of the remainder in Taylor’s Theorem (see Theorem 5.15 of Rudin’s book) to compare \( \log(1 + x) \) with the Taylor polynomials of degrees 1 and 2.

(b) Roughly how large must \( m \) be so that \( n = 10^m \) satisfies \( s_n > 100? \)

Solution (sketch). The proof above gives \( 0 < s_n - \log(n + 1) < 1 \) for all \( n \in \mathbb{Z}_{>0} \). Therefore \( s_n \in (\log(n + 1), 1 + \log(n + 1)) \). We have \( \log(n + 1) \geq 100 \) if and only if \( n > \exp(100) - 1 \). So it suffices to take

\[
m = \log_{10}(\exp(100)) = 100 \log_{10}(e) \approx 43.43.
\]

This completes the solution.

Problem 8.10: Prove that

\[
\sum_{p \text{ prime}} \frac{1}{p}
\]
diverges.

Hint: Given \( N \), let \( p_1, p_2, \ldots, p_k \) be those primes that divide at least one integer in \( \{1, 2, \ldots, N\} \). Then

\[
\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \leq \exp \left( \sum_{j=1}^{k} \frac{1}{p_j} \right).
\]

The last inequality holds because \( (1 - x)^{-1} \leq \exp(2x) \) for \( 0 \leq x \leq \frac{1}{2} \).

Solution (sketch). It suffices to verify the inequalities because \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. For the first inequality, let \( m \) be the largest power of any prime appearing in the prime factorization of any integer in \( \{1, 2, \ldots, N\} \). Let \( S \) be the set of all strictly positive integers whose prime factorization involves only the primes \( p_1, p_2, \ldots, p_k \), and in which no prime appears with multiplicity greater than \( m \). Then one checks that

\[
\prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots + \frac{1}{p_j^m}\right) = \sum_{n \in S} \frac{1}{n}.
\]

Moreover, \( \{1, 2, \ldots, N\} \subset S \). For the other inequality, since \( 0 \leq \frac{1}{p} \leq \frac{1}{2} \) for all primes \( p \), it suffices to check that \( (1 - x)^{-1} \leq \exp(2x) \) for \( 0 \leq x \leq \frac{1}{2} \). Now \( 1 + 2x \leq \exp(2x) \) for all \( x \geq 0 \) by any of a number of arguments. The inequality \( (1 - x)^{-1} \leq 1 + 2x \) for \( 0 \leq x \leq \frac{1}{2} \) is easily verified by multiplying both sides by \( 1 - x \).

Problem 8.11: Let \( f : [0, \infty) \to \mathbb{R} \) be a function such that \( \lim_{x \to \infty} f(x) = 1 \) and \( f \) is Riemann integrable on every interval \( [0, a] \) for \( a > 0 \). Prove that

\[
\lim_{t \to 0^+} t \int_{0}^{\infty} e^{-tx} f(x) \, dx = 1.
\]
Solution (sketch). Let $\varepsilon > 0$. Choose $a > 0$ such that $|f(x) - 1| < \frac{\varepsilon}{2}$ for $x \geq a$. Since $f$ is Riemann integrable on $[0, a]$, it is bounded there. Choose $M$ such that $|f(x)| \leq M$ for all $x \in [0, a]$. Choose $\delta$ so small that $\delta(M + 1) < \frac{\varepsilon}{2}$. Then $0 < t < \delta$ implies
\[
t \int_{0}^{\infty} (e^{-tx} \cdot 1) \, dx = 1
\]
and
\[
t \int_{0}^{\infty} e^{-tx} |f(x) - 1| \, dx \leq \delta \int_{0}^{a} e^{-tx} (M + 1) \, dx + t \int_{a}^{\infty} e^{-tx} \left( \frac{\varepsilon}{2} \right) \, dx
\]
\[
\leq \delta (M + 1) + \frac{\varepsilon e^{-at}}{2} < \varepsilon.
\]
This completes the solution. \qed